On the equation $F(x) = f(f(x))$

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1 The Problem

Following Ed Pegg Jr (see the section “Material added 21 November 04” on http://www.Mathpuzzle.com) we want to discuss the problem:

\[
\begin{aligned}
\{ & \text{which functions } F \text{ have a representation of the form } F(x) = f(f(x))? \\
& \text{for a given } F, \text{ how many are the solutions, and how to build them?}
\end{aligned}
\]

We will confine ourselves to work in a polynomial framework, assuming that both $F$ and $f$ are polynomials. In fact the non-polynomial framework seems out of reach because unexpected non-polynomial solutions come out very often; see later on. On the other hand, we will allow complex coefficients for our polynomials $F, f$: as often happens, the real results easily follow from the complex ones; and the complex approach gives rise to more elegant results.

Remark After reading a preliminary redaction of this paper, Ed sent me the feedback: One interesting trick to add: \((f(f(x)) - x)/(f(x) - x)\) always divides evenly thus I added a last section, “Factorizations”, absent in the previous redaction.

2 Preliminary Remarks

Let us firstly deal with a somewhat anomalous case: the function $F(x) = x + c$ has the solution $f(x) = x + c/2$. If $c \neq 0$ we will see that this is the unique solution; however, if $c = 0$, for any $a$ we can also choose $f(x) = a - x$.

We will also see that, in the polynomial framework, the case $F(x) = x$ is the unique one giving raise to infinitely many solutions. Remark that this $F$ is singular even in the non-polynomial framework: for $x \neq 0$ also the functions $f(x) = a/x$ solve. A last remark on the non-polynomial case: the rational function $F(x) = 2x/(1 - x^2)$ has, for $|x| < 1$, the transcendental solution $f(x) = \tan \left(\sqrt{2} \cdot \arctan(x)\right)$.

Let us fix some notations. For $f$ polynomial of degree $k$, the polynomial $f(f(x))$ has degree $k^2$; thus, for a suitable $k$, we can represent the given $F$ and the
unknown $f$ in the form:

$$F(x) = \sum_{j=0}^{k^2} A_j x^{k^2-j}; \quad f(x) = \sum_{j=0}^{k} a_j x^{k-j}; \quad (A_0, a_0 \neq 0);$$

remark that we used capital letters $A_j$ for the data, and small letters $a_j$ for the unknown coefficients. In other terms: we would freely choose the $k^2 + 1$ coefficients of $F$, and impose the equation $F(x) = f(f(x))$ by using the $k + 1$ coefficients of $f$. It is quite natural to expect that, if $k > 1$, this is impossible; and in fact we will see that, generally speaking, once we fix the first $k + 1$ coefficients $A_0, \ldots, A_k$ we have no more degree of freedom.

### 3 Results

Let us evaluate $f(f(x))$ modulo the lower order terms, say the terms of order strictly less than $k^2 - k$; we will use the symbol “$\cong$” to denote that some lower order terms have been suppressed. Because of

$$\begin{align*}
  f(f(x)) &\cong a_0 \cdot [f(x)]^k + a_1 \cdot [f(x)]^{k-1}; \\
  a_0 \cdot [f(x)]^k &\cong a_0 \cdot \left(a_0 \cdot x^k\right)^k + k \cdot [a_0 \cdot x^k]^{k-1} \cdot [a_1 \cdot x^{k-1} + \cdots + a_k]; \\
  a_1 \cdot [f(x)]^{k-1} &\cong a_1 \left[a_0 \cdot x^k\right]^{k-1}
\end{align*}$$

we get:

$$f(f(x)) \cong a_0^{k+1} \cdot x^{k\cdot k} + k \cdot [a_0 \cdot x^k]^{k-1} \cdot [a_1 \cdot x^{k-1} + \cdots + a_k] + a_1 \left[a_0 \cdot x^k\right]^{k-1}$$

The comparison between the leading coefficients of $F(x)$ and $f(f(x))$ gives:

$$A_0 = a_0^{k+1}; \quad A_j = k \cdot a_0^k \cdot a_j \text{ for } j = 1, \ldots, k-1; \quad A_k = k \cdot a_0^k \cdot a_k + a_1 \cdot a_0^{k-1}.$$

In particular, from $a_0^{k+1} = A_0$, we have $k + 1$ possible choices for $a_0$ (recall that $A_0 \neq 0$). Once a value for $a_0$ has been fixed:

- if $k = 0$ we finished;
- if $k = 1$, we need $a_1 = A_1 / (a_0 + 1)$, to be discussed later on;
- if $k > 1$ we must choose:

  $$a_j = A_j / (k \cdot a_0^k) \text{ for } j = 1, \ldots, k-1;$$

  $$a_k = (A_1 - a_1 \cdot a_0^k) / (k \cdot a_0^k) = (A_k - A_1 / k) / (k \cdot a_0^k)$$

  the last formula following from the already known form of $a_1$.

The case $k = 1$, as already remarked, deserves a surprise: for $A_0 \neq 1$ we simply need to choose $a_1 = A_1 / (a_0 + 1)$; the same formula remains true if $A_0 = 1$ and we choose $a_0 = 1$; however, for $A_0 = 1$, if we want to choose $a_0 = -1$, we need $A_1 = 0$; then any value for $a_1$ is allowed.

Let us summarize the results in the complex framework. Searching for representable polynomials $F(x)$ of degree $k$, we have:
• If $k < 2$, for any choice of $F(x)$ other than $x + c$, there exist exactly $k + 1$ solutions.

• If $k \geq 2$ we can freely choose the first $k + 1$ coefficients of $F$; this generates $k + 1$ functions $f$ that automatically select their own “lower order part” for $F$.

Let us show how the last claim works by discussing the original example $F(x) = x^4 - 4x^3 + 8x + 2$ proposed by Ed Pegg. Our formulas force $f(x) = \omega \cdot x^2 - 2\omega \cdot x + 1 - 2\omega$, where $\omega$ denotes a third root of the unit. Accordingly we get $f(f(x)) = x^4 - 4x^3 + 8x + 5 - 3\omega$; thus we need $\omega = 1$, say $f(x) = x^2 - 2x - 1$.

A last remark concerns the real framework: let $F(x)$ be a real polynomial of degree $k$, other than $x + c$. Among the $k + 1$ complex-coefficients polynomials $f$ we have built, the number of real $f$ is just one if $k$ is odd; and, for even $k$, is two or none, according to $A_0 > 0$ or $A_0 < 0$.

4 Factorizations

A somewhat different approach can be based on factorizations results; e.g. let us remark that:

\[ F'(x)/f'(x) \text{ always divides evenly} \]

as obvious because of $F''(x) = f(f(x))' = f'(f(x)) \cdot f'(x)$.

**Remark** Let $x_0$ be a (real or complex) value such that $f(x_0) = x_0$. Then:

• $F'(x_0) = f'(x_0)^2$.

• $F''(x_0) = f''(x_0)f'(x_0)^2 + f'(x_0)f''(x_0)$

• For $k > 2$ any term appearing in $F^{(k)}(x_0)$ contains a factor like $f^{(j)}(x_0)$ for a $j \geq 2$.

Now let us set $G(x) := f(f(x)) - x$ and $g(x) := f(x) - x$. For any $x_0$ with $g(x_0) = 0$ one has $G(x_0) = 0$; if one has also $g'(x_0) = 0$, say $f'(x_0) = 1$ then (see the previous remark) $G'(x_0) = F'(x_0) - 1 = 0$; always due to the previous remark, if $g''(x_0) = 0$, say $f''(x_0) = 0$, then $G''(x_0) = 0$; and so on.

In other words, any (real or complex) root of $g$ is also a root of $G$ with at least the same multiplicity; this implies that $g(x)$ is a factor of $G(x)$; say:

\[ (f(f(x)) - x)/(f(x) - x) \text{ always divides evenly} \]

Let us see how these properties can be used, by treating again the example $F(x) = x^4 - 4x^3 + 8x + 2$: we confine ourselves to discuss the real case, thus the leading coefficients of $f$ and $f'$ must be 1 and 2, instead of $\omega$ and $2\omega$.

Because of $F'(x) = 4(x-1)(x-1 - \sqrt{3})(x-1 + \sqrt{3})$, for $f'(x)$ we have to choose one among $2(x-1)$, $2(x-1 - \sqrt{3})$ and $2(x-1 + \sqrt{3})$.

Because of $F(x) - x = (x - 2)(x + 1)(x - 2 + \sqrt{3} + 1)(x - 2 - \sqrt{3} - 1)$ we have a priori six possible choices for the second degree polynomial $f(x) - x$; but only the choice $f(x) - x = (x - 2 + \sqrt{3} + 1)(x - 2 - \sqrt{3} - 1) = x^2 - 3x - 1$ is compatible with the formulas we got for $f'(x)$; thus we end up with the already found solution $f(x) = x^2 - 2x - 1$. 

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