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The existence of $(v, 6, \lambda)$-perfect Mendelsohn designs with $\lambda > 1$

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Abstract The basic necessary conditions for the existence of a (v, k, λ)-perfect Mendelsohn design (briefly (v, k, λ)-PMD) are $v \geq k$ and $\lambda v(v-1) \equiv 0 \pmod{k}$. These conditions are known to be sufficient in most cases, but certainly not in all. For $k = 3, 4, 5, 7$, very extensive investigations of (v, k, λ)-PMDs have resulted in some fairly conclusive results. However, for $k = 6$ the results have been far from conclusive, especially for the case of $\lambda = 1$, which was given some attention in papers by Miao and Zhu [34], and subsequently by Abel et al. [1]. Here we investigate the situation for $k = 6$ and $\lambda > 1$. We find that the necessary conditions, namely $v \geq 6$ and $\lambda v(v-1) \equiv 0 \pmod{6}$ are sufficient except for the known impossible cases $v = 6$ and either $\lambda = 2$ or λ odd.

Keywords Perfect Mendelsohn design · Incomplete Perfect Mendelsohn design · Holey Perfect Mendelsohn design · GDD · PBD

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1 Introduction

A set of k elements $\{a_1, a_2, \ldots, a_k\}$ is said to be cyclically ordered by $a_1 < a_2 < \cdots < a_k < a_1$ and the pair a_i, a_{i+t} are said to be *t-apart* in a cycle k-tuple $(a_1, a_2, \ldots, a_k)$ where $i + t$ is taken modulo k.

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Let v, k and λ be positive integers. A (v, k, λ)-*Mendelsohn design*, denoted briefly by (v, k, λ)-MD, is a pair $(X, \mathbf{B})$ where X is a v-set (of *points*) and $\mathbf{B}$ is a collection of cyclically ordered k-subsets of X (called *blocks*) such that every ordered pair of points of X are consecutive in exactly λ blocks of $\mathbf{B}$. If for all $t = 1, 2, \ldots, k-1$, every ordered pair of points of X are t-apart in exactly λ blocks of $\mathbf{B}$, then the (v, k, λ)-MD is called a *perfect* design and denoted briefly by (v, k, λ)-PMD.

If we ignore the cyclic order of the points, then a $(v, k, 1)$-PMD becomes a *balanced incomplete block design* with parameters v, k and $\lambda = k-1$, briefly denoted by $(v, k, k-1)$-BIBD. Therefore, we can consider a perfect Mendelsohn design as a generalization of balanced incomplete block designs. Mendelsohn [32] first introduced the concept of a perfect cyclic design. This concept has been further studied by various authors, including Hsu and Keedwell [28], where the designs were called perfect Mendelsohn designs. We have since adopted this terminology.

It is easy to see that the number of blocks in a (v, k, λ)-PMD is $\lambda v(v-1)/k$. Consequently, a basic necessary condition for the existence of a (v, k, λ)-PMD is the following:

$$\lambda v(v-1) \equiv 0 (\text{mod } k).$$

For $k = 3, 4$, the problem of existence of (v, k, λ)-PMDs has been completely settled. We have the following conclusive results.

Theorem 1.1

1. [8, 31] *The necessary conditions for existence of a* $(v, 3, \lambda)$*-PMD, namely,* $v \geq 3$ *and* $\lambda v(v-1) \equiv 0$ *(mod* 3*), are also sufficient, except for the non-existing* $(6, 3, 1)$*-PMD.*
2. [17, 21, 33, 38] *The necessary conditions for existence of a* $(v, 4, \lambda)$*-PMD, namely,* $v \geq 4$ *and* $\lambda v(v-1) \equiv 0$ *(mod* 4*), are also sufficient, except for* λ *odd when* $v = 4$*, and* $\lambda = 1$ *when* $v = 8$.

For $(v, 5, \lambda)$-PMDs, investigations by various authors have resulted in the following almost conclusive result.

Theorem 1.2 [11, 12, 14, 15, 18] *The necessary condition for the existence of a* $(v, 5, \lambda)$*-PMD, namely,* $\lambda v(v-1) \equiv 0$ *(mod* 5*), is also sufficient, except for* $\lambda = 1, v \in \{6, 10\}$*, and possibly for* $\lambda = 1, v \in \{15, 20\}$.

The results for $k = 6$ are not so conclusive. In particular, for $k = 6$ and $\lambda = 1$, the necessary existence condition is $v \equiv 0, 1, 3$ or 4 (mod 6) and $v \geq 6$, but only the case $v \equiv 1$ (mod 6) has been resolved completely here. It is known that a (6, 6, 1)-PMD does not exist. We also have one new nonexistence result for $\lambda = 1$:

Theorem 1.3 *There is no* $(10, 6, 1)$*-PMD.*

Proof If a (10, 6, 1)-PMD did exist, then breaking up each ordered block $(a1, a2, a3, a4, a5, a6)$ into 2 smller blocks $(a1, a3, a5)$ and $(a2, a4, a6)$ would give a (10,6,5) BIBD with a (10,3,2) BIBD as a nested subdesign. However, Hishida et al. [24] showed there is no such BIBD, so there is no (10,6,1)-PMD either. □

The existence of $(v, 6, 1)$-PMDs was investigated by Miao and Zhu [34] and by Abel et al. [1]. Their results together with the above nonexistence result for $v = 10$ are summarized in the following theorem.

Theorem 1.4 [1, 34] *The necessary conditions for the existence of a* $(v, 6, 1)$*-PMD, namely* $v \equiv 0, 1, 3$ *or* 4 *(mod* 6*) and* $v > 6$*, are sufficient except for the cases* $v = 6, 10$*, and possibly for the following cases:*

1. $v \equiv 0 (mod\ 6)$ *and* $v \in \{12, 18, 24, 30, 48, 54, 60, 72, 84, 90, 96, 102, 108, 114, 132, 138, 150, 162, 168, 180, 192, 198\}$.
2. $v \equiv 3\ (mod\ 6)$ *and either* $v \in \{207, 213, 219, 237, 243, 255, 297, 375, 411, 435, 453, 459, 471, 489, 495, 513, 519, 609, 615, 621, 657\}$ *or v is in one of the following intervals:* $[9, 135]$, $[153, 183]$.
3. $v \equiv 4 (mod\ 6)$ *and either* $v \in \{16, 22, 34\}$ *or v is in the interval* $[52, 148]$.

In addition to the above, the existence of $(v, 6, 3)$-PMDs was examined in [36] and in Theorem 10.2 of [1]. The next theorem provides a summary of these known results.

Theorem 1.5 [1, 36] *A* $(v, 6, 3)$*-PMD exists for all integers* $v \geq 6$ *except possibly for* $v \in \{6, 10, 12, 16, 18, 22, 24, 30, 33, 34, 39, 45, 48, 51, 54, 60\}$.

In contrast to the case of $k = 6$, the problem of existence of $(v, 7, \lambda)$-PMDs has recently been reduced to relatively few possible exceptions. Five such new PMDs are given in Lemma 4.1. Combining these with the known results in [2, 3, 19], we now have the following theorem.

Theorem 1.6 [2, 3, 19] *Necessary conditions for the existence of a* $(v, 7, \lambda)$*-PMD are* $v \geq 7$ *if* $\lambda \equiv 0\ (mod\ 7)$*, or* $v \equiv 0$ *or* $1\ (mod\ 7)$, $v \geq 7$*, if* $\lambda \not\equiv 0\ (mod\ 7)$. *These conditions are sufficient except possibly for* $\lambda = 1$ *and* $v \in \{14, 15, 21, 22, 28, 35, 36, 42, 70, 84, 98, 99, 126, 140, 141, 147, 148, 154, 182, 183, 196, 238, 245, 273, 294\}$.

In this paper, we shall attempt to fill the apparent gap in the above existence results for $(v, 6, \lambda)$-PMDs. Here we focus our attention on the problem of existence of $(v, 6, \lambda)$-PMDs for all $\lambda > 1$. We establish for all such λ that the necessary conditions, namely $v \geq 6$ and $\lambda v(v - 1) \equiv 0 \pmod 6$, are sufficient for the existence of a $(v, 6, \lambda)$-PMD, except for the pairs (v, λ), where $v = 6$ and either $\lambda = 2$ or λ is odd. More formally, we prove the following theorem:

Theorem 1.7 *Necessary conditions for existence of a* $(v, 6, \lambda)$*-PMD are* (1) $v \geq 6$*, and* (2) $v \equiv 0$ *or* $1\ (mod\ 3)$ *if* $\lambda \not\equiv 0\ (mod\ 3)$. *For* $\lambda > 1$*, these are sufficient except for* $v = 6$ *and either* $\lambda = 2$ *or* λ *odd.*

It is sufficient to establish the preceding theorem just for $\lambda = 2, 3$ when $v \neq 6$, and for $\lambda = 4, 6$ when $v = 6$, since $(v, 6, \lambda)$-PMDs with larger λ can then be handled using the following obvious lemma:

Lemma 1.8 *If there exist both a* (v, k, λ_1)*-PMD and a* (v, k, λ_2)*-PMD, then a* $(v, k, s\lambda_1 + t\lambda_2)$*-PMD exists for all non-negative integers s, t.*

2 Auxiliary designs

In order to establish our main result, we shall employ both direct and recursive constructions. In this section, we shall define some terminology and describe some of the auxiliary designs to be used in our constructions. For more detailed information on some of these related combinatorial structures, the reader is referred to [22, 35].

A *holey perfect Mendelsohn design* (HPMD) with block size k is an order triple $(X, Y, \mathbf{A})$ such that X is a set of points, $Y = \{Y_1, Y_2, \ldots, Y_n\}$ is a partition of X, and $\mathbf{A}$ is a set of (cyclically ordered) blocks of size k with the following properties:

(1) For any integer t ($1 \leq t \leq k-1$) and any two points z_1, z_2 from different sets Y_i, Y_j, there are exactly λ blocks A_m in $\mathbf{A}$ such that z_1 and z_2 appear t-apart is A_m;
(2) No two points in any set Y_i appear together in any block from $\mathbf{A}$.

Each Y_i ($1 \leq i \leq n$) is called a *hole* (or *group*) of the design and the multiset $\{|Y_1|, |Y_2|, \ldots, |Y_n|\}$ is called the *type* of the design. We denote such a design by (v, k, λ)-HPMD (or briefly (k, λ)-HPMD) and use an "exponential" notation to describe its type in general: a type $1^{r1}2^{r2}3^{r3}\ldots$ denotes r_1 holes of size 1, r_2 holes of size 2, etc.

A (v, k, λ)-HPMD of type 1^v is essentially a (v, k, λ)-PMD. Also, a $(v, k, 1)$-HPMD of type $1^{v-n}n^1$ is called an *incomplete a perfect Mendelsohn design*, denoted by (k, λ)-IPMD(v, n).

A *pairwise balanced design* (PBD) is a pair $(X, \mathbf{A})$ such that X is a set of elements (called *points*), and $\mathbf{A}$ is a set of subsets (called *blocks*) of X, each of cardinality at least two, such that every unordered pair of points is contained in a unique block in $\mathbf{A}$. If v is a positive integer and K is a set of positive integers, each of which is not less than 2, then we say that $(X, \mathbf{A})$ is a (v, K)-PBD if $|X| = v$, and $|A| \in K$ for every $A \in \mathbf{A}$. The integer v is called the *order* of the PBD. Using this notation, we can define a BIBD $B(k, 1; v)$ to be a $(v, \{k\})$-PBD. We shall denote by $B(K)$ the set of all integers v for which there exists a (v, K)-PBD. For convenience, we define $B(k_l, k_2, \ldots, k_r)$ to be the set of all integers v such that there is a $(v, \{k_1, k_2, \ldots, k_r\})$-PBD.

A (K, λ) *group divisible design* (GDD) is a triple $(X, \mathbf{G}, \mathbf{B})$ which satisfies the following properties:

1. $\mathbf{G}$ is a partition of a set X (of *points*) into subsets called *groups*,
2. $\mathbf{B}$ is a set of subsets of X (called *blocks*) with sizes from K such that any group and any block contain at most one common point.
3. every pair of points from distinct groups occurs in λ blocks.

The parameter λ is often not specified if it equals 1. Also if K consists of a single value, k, we often write k instead of $\{k\}$.

The *group-type* (or *type*) of the GDD is the multiset $\{|G|: G \in \mathbf{G}\}$. As with HPMDs we use an "exponential" notation to describe group-type.

A *transversal design*, $TD_\lambda(k, n)$ (or TD(k, n) if $\lambda = 1$) is a (k, λ)-GDD with group type n^k. It is well known that a TD(k, n) is equivalent to $k-2$ mutually orthogonal Latin squares (MOLS) of order n. It is also well known that the existence of a $(v, k, 1)$-PMD implies the existence of a TD(k, v), with an automprphism which cyclically permutes its groups. We have the following existence results for TDs with $k = 6$:

Lemma 2.1

1. [5] *A TD$(6, m)$ exists for all integers $m > 4$ except for $m = 6$ and possibly for $m \in$* $\{10, 14, 18, 22\}$.
2. [26] *If $\lambda > 1$, a $TD_\lambda(6, m)$ exists for all positive integers m.*

3 Recursive construction methods

3.1 Fill-in-hole

One importance of HPMDs and IPMDs is that their holes can frequently be filled to give a PMD. This simple but very effective approach has been frequently used in the construction of several other combinatorial structures such as GDDs and PBDs. See, for instance, ([11, 17, 19, 34, 38]) for other examples of this approach for PMDs.

Construction 3.1 *If a* (k, λ)*-IPMD*(v, h) *and an* (h, k, λ)*-PMD both exist, then so does a* (v, k, λ)*-PMD.*

Construction 3.2 *Suppose a* (k, λ)*-HPMD of type* $(h_1, h_2, \ldots, h_n)$ *exists. Suppose also* $w \geq 0$ *and there exist a* (k, λ)*-IPMD*$(h_i + w, w)$ *for* $1 \leq i \leq n - 1$*, plus an* $(h_n + w, k, \lambda)$*-PMD. Then a* (v, k, λ)*-PMD exists for* $v = \sum h_i + w$*.*

3.2 Weighting

In recursive constructions of GDDs and PBDs, the "weighting" technique and Wilson's Fundamental GDD construction [35] are frequently used. Similar techniques are also available for constructing HPMDs. Here, we start with a master GDD and use HPMDs as ingredients for inflation. For more details on these techniques see [15, 19, 34].

Construction 3.3 (Weighting) *Suppose* $(X, \mathbf{G}, \mathbf{B})$ *is a GDD with index* λ_1 *and w is a function from X to* $Z^+ \cup \{0\}$*. Suppose there exists a* (k, λ_2)*-HPMD of type* $\{w(x): x \in B\}$ *for every* $B \in \mathbf{B}$*. Then there exists a* $(k, \lambda_1\lambda_2)$*-HPMD of type* $\sum_{x \in G} w(x)$: $G \in \mathbf{G}$.

In particular, if the given GDD is a PBD, and all points are given weight 1, we have

Construction 3.4 *If there exist a* (v, K, λ_1)*-PBD and a* (t, k, λ_2)*-PMD for all* $t \in K$*, then there exists a* $(t, k, \lambda_1\lambda_2)$*-PMD.*

It is also possible to start with an HPMD and inflate using TDs as in the following construction:

Construction 3.5 *If a* (k, λ)*-HPMD of type* $(v_1, v_2, \ldots, v_h)$ *and a* $TD_\mu(k, m)$ *both exist, then a* $(k, \lambda\mu)$*-HPMD of type* $(mv_1, mv_2, \ldots, mv_h)$ *exists.*

Proof Let $V_1, V_2, \ldots, V_h$ be the holes in the (k, λ)-HPMD of type $(v_1, v_2, \ldots, v_h)$, and let the TD(k, m) be over $I_k \times M$ where $I_k = \{1, 2, \ldots, k\}$ and M is a set of size m. The size m groups in this TD will be taken as $\{x\} \times M$ for $x \in I_k$. Now for each block $(b_1, b_2, \ldots, b_k)$ in the k-HPMD$(v_l, v_2, \ldots, v_h)$ with holes and each block $\{(1, a_1), (2, a_2), \ldots, (k, a_k)\}$ in the TD(k, m) we form a block $\{(b_1, a_1), (b_2, a_2), \ldots, (b_k, a_k)\}$. It is readily checked that this gives the required HPMD with holes $V_i \times M$ for $1 \leq i \leq h$. □

4 Some direct constructions

Finite fields and abelian groups play an important role in our direct constructions. For most of our direct constructions, we adapt the familiar method using difference sets as in the construction of BIBDs, where we use finite abelian groups to generate the set of blocks for a given design. That is, instead of listing all the blocks of the design, we shall list a set of base blocks and generate the others by adding elements of a given abelian group G to them. Also when indicated, some base blocks should be multiplied by certain values before developing them additively over G.

Before proceeding with the case $k = 6$, we shall provide five new important constructions for Theorem 1.6.

Lemma 4.1 *There exist an* $(18, 7, 7)$*-PMD, and* $(42, 7, \lambda)$*-PMDs for* $\lambda \in \{2, 3, 5, 9\}$*.*

Proof For an (18, 7, 7)-PMD, take the point set as $Z_{17} \cup \{\infty\}$. Multiply the first six blocks below by 1 only, and the last six by 1, −1. Then develop the resulting 18 blocks (mod 17):

$$\begin{array}{lll}(0, 2, 7, 16, 5, 11, 1), & (0, 1, 2, 15, 7, 6, \infty), & (0, 8, 11, 13, 10, 14, \infty),\\ (0, 1, 15, 11, 10, 13, \infty), & (0, 15, 13, 8, 16, 10, \infty), & (0, 4, 16, 10, 3, 8, \infty),\\ (0, 7, 4, 2, 1, 9, 10), & (0, 7, 2, 16, 14, 13, 1), & (0, 9, 11, 2, 4, 6, 13),\\ (0, 5, 16, 7, 13, 2, 12), & (0, 3, 12, 8, 4, 11, 5), & (0, 3, 16, 10, 7, 11, \infty).\end{array}$$

This design possesses one unusual feature: for any t, the multiset of t-apart differences in the first six blocks remains invariant when multiplied by −1, but this does not apply to any proper subset of these six blocks. Another example of this comes from the first six base blocks in the (34, 6, 3)-PMD given later in Lemma 4.8.

For a (42, 7, 2)-PMD, take the point set as $Z_{41} \cup \{\infty\}$. Multiply the six blocks below by 1, −1 and develop the resulting 12 blocks (mod 41):

$$(0, 2, 3, 15, 25, 1, 23),\quad (0, 2, 18, 24, 13, 7, 28),\quad (0, 31, 16, 37, 33, 4, 34),$$
$$(0, 7, 31, 35, 22, 4, 36),\quad (0, 1, 23, 9, 35, 30, 33),\quad (0, 9, 34, 20, 28, 25, \infty).$$

For a (42, 7, 3)-PMD, take the point set as $Z_{41} \cup \infty$. Multiply the first six blocks below by 1 only and the last six by 1, −1. Then develop the resulting 18 blocks (mod 41):

$$\begin{array}{lll}(0, 1, 5, 12, 33, 24, 3), & (0, 40, 14, 31, 12, 17, 30), & (0, 3, 36, 10, 8, 14, 11),\\ (\infty, 40, 36, 29, 8, 17, 38), & (\infty, 1, 27, 10, 29, 24, 11), & (\infty, 38, 5, 31, 33, 27, 30),\\ (0, 12, 26, 36, 20, 16, 2), & (0, 16, 14, 9, 26, 3, 11), & (0, 5, 40, 17, 7, 34, 6),\\ (0, 11, 4, 17, 26, 25, 22), & (0, 33, 10, 20, 19, 39, 24), & (0, 7, 19, 28, 40, 3, 25).\end{array}$$

For $(42, 7, \lambda)$ PMDs with $\lambda \in \{5, 9\}$, we can apply Lemma 1.8, combining a (42, 7, 3)-PMD with either one or three copies of a (42, 7, 2)-PMD. □

4.1 $(6, 6, \lambda)$-PMDs

A computer search by Hantao Zhang [37] has established that no (6, 6, 2)-PMD exists. In this section, we prove that there is no $(6, 6, \lambda)$-PMD with λ odd. We also give $(6, 6, \lambda)$-PMDs for λ even and ≥ 4.

Non-existence of a $(6, 6, \lambda)$-PMD with λ odd, follows from the following more general nonexistence result:

Theorem 4.2 *If $k > 2$ and $k \equiv 2 \pmod 4$, then there is no (k, k, λ)-PMD with λ odd.*

Proof Suppose such a (k, k, λ)-PMD exists with λ odd. We partition each ordered block $B = (b_l, b_2, b_3, \dots, b_k)$ PMD into two smaller blocks of size $k/2$: $(b_1, b_3, b_5, \dots, b_{k-1})$ and $(b_2, b_4, b_6, \dots, b_k)$. From the t-apart properties of the points in the blocks of a PMD for t even (if $k > 2$), this partitioning gives rise to a resolvable $(k, k/2, \lambda)$-PMD, which would imply the existence of a resolvable $(k, k/2, \lambda(k/2 - 1))$-BIBD. However, it is known that no such resolvable BIBD can exist for $k \equiv 2 \pmod 4$, $k > 2$ and λ odd [25], a contradiction. □

Theorem 4.3 *There exists a $(6, 6, \lambda)$-PMD for all even $\lambda \geq 4$.*

Proof For $\lambda = 4$ and 6, we take the point set as $Z_5 \cup \{\infty\}$. For $\lambda = 4$, develop the following blocks (mod 5): $(0, t, 3t, 2t, 4t, \infty)$ for $1 \leq t \leq 4$. For $\lambda = 6$, the blocks to be developed (mod 5) are $(0, 2, 1, 3, 4, \infty)$, $(0, 1, 2, 4, 3, \infty)$, $(0, 1, 3, 4, 2, \infty)$, $(0, 2, 4, 3, 1, \infty)$, $(0, 3, 2, 1, 4, \infty)$, and $(0, 4, 2, 3, 1, \infty)$. Designs for larger even values of λ can now be obtained by using multiple copies of these two designs. □

4.2 Direct constructions for $\lambda = 2$

We start by providing a number of small designs constructed directly. Most of these are difference family type constructions.

Lemma 4.4 *If* $v \in \{9, 10, 12, 15, 16, 18, 21, 22, 24, 27, 28, 30, 33, 34, 39, 45\}$ *then a* $(v, 6, 2)$*-PMD exists.*

Proof In each case we give the point set as either Z_v (or $Z_{v-1} \cup \{\infty\}$) and provide a number of base blocks to be developed (mod v) or (mod $v-1$). For $v = 9$ and 10, add only multiples of 2 (mod v or $v-1$) to the given blocks, but for the others, all blocks should be fully developed (mod v or $v-1$). Also, where indicated, multiply the given blocks by the values listed before developing them. □

v	Point set	Base blocks
9	$Z_8 \cup \{\infty\}$	(0, 6, 2, 5, 4, 3), (5, 1, 7, 6, 3, 4), (0, 2, 4, 3, 1, ∞), (5, 7, 1, 4, 2, ∞), (0, 1, 6, 2, 5, ∞), (5, 6, 3, 7, 2, ∞).
10	Z_{10}	(0, 2, 6, 5, 8, 1), (5, 3, 7, 6, 1, 4), (0, 2, 8, 6, 3, 9), (5, 1, 3, 7, 8, 6), (0, 7, 4, 9, 1, 6), (5, 2, 3, 8, 4, 7).
12	$Z_{11} \cup \{\infty\}$	(0, 2, 5, 10, 9, ∞), (0, 4, 5, 7, 2, ∞), (0, 5, 4, 1, 10, 7), (0, 6, 7, 5, 8, 4).
15	$Z_{14} \cup \{\infty\}$	(0, 2, 7, 8, 3, ∞), (0, 4, 9, 12, 11, ∞), (0, 6, 4, 11, 7, 1), (0, 6, 10, 5, 2, 4), (0, 11, 12, 5, 13, 2).
16	Z_{16}	(0, 4, 10, 9, 12, 3), (0, 1, 6, 4, 15, 9), (0, 11, 8, 2, 1, 13), (0, 4, 12, 1, 10, 8), (0, 2, 4, 5, 1, 7).
18	$Z_{17} \cup \{\infty\}$	(0, 1, 7, 12, 3, ∞), (0, 10, 13, 16, 14, ∞), (0, 8, 10, 2, 15, 11), (0, 2, 9, 6, 7, 1), (0, 4, 8, 7, 12, 5), (0, 7, 2, 13, 10, 8).
21	$Z_{20} \cup \{\infty\}$	(0, 6, 9, 18, 15, ∞), (0, 10, 1, 18, 5, ∞), (0, 8, 2, 4,14, 15), (0, 2, 10 16, 11, 7), (0, 4, 16, 8, 3, 1), (0, 4, 2, 3, 10, 1), (0, 14, 17, 13, 2, 7).
22	Z_{22}	(0, 6, 9, 11, 18, 5), (0, 16, 12, 1, 2, 3), (0, 8, 4, 2, 14, 13), (0, 4, 20, 18, 1, 15), (0, 8, 18, 1, 15, 3), (0, 2, 6, 19, 12, 11), (0, 6, 21, 2,14, 5).
24	$Z_{23} \cup \{\infty\}$	(0,1, 4, 15, 7, ∞), (0, 4, 13, 11, 22, 16), (0, 2, 8, 13, 21, 7) (0, 10, 6, 9, 8, 13). (Multiply these 4 blocks by 1 amd −1)
27	$Z_{26} \cup \{\infty\}$	(0, 19, 18,14, 12, ∞), (0, 11, 10, 16, 14, ∞), (0, 4, 8, 20, 9, 23), (0, 2, 12, 9, 23, 5), (0, 6, 24, 25, 8, 23), (0, 8, 2, 9, 10, 15), (0, 12, 2, 18, 15, 17), (0, 10, 17, 4, 22, 13), (0, 20, 25, 16, 12, 5).
30	$Z_{29} \cup \{\infty\}$	(0, 12, 21, 13, 10, ∞), (0, 1, 3, 18, 7, 22), (0, 2, 5, 1, 11, 17), (0,1, 9, 4, 22, 6), (0, 9, 14, 7, 23, 19). (Multiply these 5 blocks by 1 and −1)
33	$Z_{32} \cup \{\infty\}$	(0, 1, 31, 16, 15, 17), (0, 7, 12, 25, 10, ∞), (0, 1, 4, 8, 12, 10), (0, 6, 31, 20, 4, 9), (0, 9, 17, 28, 8, 22), (0, 3, 21, 13, 26, 6). (Multiply the last 5 blocks by 1 and −1)
34	Z_{34}	(0, 1, 33, 2, 4, 8), (0, 5, 13, 27, 24, 19), (0, 6, 25, 4, 22, 12), (0, 7, 16, 10, 30, 17), (0, 10, 21, 12, 5, 1), (0, 1, 24, 20, 32, 18), (0, 3, 21, 26, 20, 12), (0, 2, 11, 18, 24, 21), (0, 14, 12, 33, 16, 5), (0, 4, 28, 21, 33, 9), (0, 8, 7, 23, 4, 15).
39	$Z_{38} \cup \{\infty\}$	(0, 1, 37, 19, 18, 20), (0, 15, 24, 32, 10, ∞), (0, 1, 4, 8, 12, 30), (0, 10, 31, 14, 8, 35), (0, 9, 35, 20, 6, 19), (0, 7, 13, 23, 36, 12), (0, 7, 2, 13, 35, 33). (Multiply the last 6 blocks by 1 and −1)
45	$Z_{44} \cup \{\infty\}$	(0, 1, 43, 22, 21, 23), (0, 6, 15, 34, 8, ∞), (0, 1, 4, 8, 12, 22), (0, 6, 30, 17, 10, 37), (0, 3, 17, 28, 12, 29), (0, 5, 13, 31, 2, 14), (0, 13, 38, 29, 34, 32), (0, 11, 35, 14, 6, 16). (Multiply the last 7 blocks by 1 and −1)

Lemma 4.5 *If* $v \in \{48, 54, 60\}$*, there exists a* $(6, 2)$*-IPMD*$(v, 7)$ *and hence also, a* $(v, 6, 2)$*-PMD.*

Proof In each case we take the point set as $Z_{v-7} \cup \{\infty_1, \infty_2, \ldots, \infty_7\}$. Multiply the given base blocks by $1, -1$, and then develop them (mod $v - 7$). Finally, form a (7, 6, 2)-PMD on the infinite points.

v	Base blocks
48	(0, 1, 15, 34, 10, 20), (0, 2, 26, 33, 25, 37), (0, 3, 35, 12, 16, ∞_1), (0, 5, 27, 22, 7, ∞_2), (0, 6, 36, 4, 3, ∞_3), (0, 8, 23, 21, 34, ∞_4), (0, 10, 37, 40, 20, ∞_5), (0, 12, 28, 35, 29, ∞_6), (0, 16, 3, 14, 32, ∞_7).
54	(0, 23, 2, 33, 18, 30), (0, 2, 31, 15, 45, 44), (0, 4, 11, 31, 33, 10), (0, 6, 12, 45, 9, ∞_1), (0, 11, 46, 7, 22, ∞_2), (0, 7, 11, 24, 15, ∞_3), (0, 9, 10, 37, 15, ∞_4), (0, 13, 21, 26, 1, ∞_5), (0, 14, 17, 22, 41, ∞_6), (0, 18, 28, 9, 35, ∞_7).
60	(0, 33, 48, 21, 17, 29), (0, 23, 5, 29, 33, 31), (0, 2, 41, 1, 38, 44), (0, 5, 18, 23, 40, 32), (0, 1, 47, 33, 23, ∞_1), (0, 9, 16, 47, 14, ∞_2), (0, 11, 39, 45, 7, ∞_3), (0, 16, 28, 9, 27, ∞_4), (0,19, 30, 2, 34, ∞_5), (0, 1, 9, 12, 29, ∞_6), (0, 23, 50, 7, 10, ∞_7).

□

The next lemma provides two useful IPMDs. The first is one of our main ingredients required for the recursive constructions in Lemma 5.2.

Lemma 4.6 *A* $(6, 2)$*-IPMD*$(11, 2)$ *and a* $(6, 1)$*-IPMD*$(51, 10)$ *both exist.*

Proof For (6, 2)-IPMD(11, 2), we take the point set as $Z_9 \cup \{\infty_1, \infty_2\}$, and develop the following four blocks (mod 11): (0, 5, 1, 3, 4, ∞_1), (0, 1, 7, 2, 6, ∞_1), (0, 8, 2, 5, 7, ∞_2), (0, 7, 4, 2, 1, ∞_2).

For a (6, 1)-IPMD(51, 10), the point set is $Z_{41} \cup \{\infty_1, \infty_2, \ldots, \infty_{10}\}$. Multiply the two blocks (0, 1, 6, 13, 33, ∞_1), (0, 26, 10, 25, 12, ∞_6) by 37^i for $0 \leq i \leq 4$, and replace ∞_1, ∞_6 by ∞_{1+i} and ∞_{6+i} when multiplying by 37^i. Then develop the resulting 10 blocks (mod 41). □

Lemma 4.7 *If* $v \in \{52, 58, 69, 75, 87\}$ *a* $(v, 6, 2)$*-PMD exists.*

Proof For $v = 52, 58$, (6, 1)-HPMDs of types $1^{43}9^1$ and $7^7 9^1$ are given in Lemmas A.7 and A.8 of [1]; take 2 copies of these and fill in the relevant holes with $(t, 6, 2)$-PMDs for $t = 7, 9$. For $v = 87$, apply Construction 3.5, inflating the (6, 1)-HPMD($3^{13}4^1$) in Lemma A.5 of [1] with a $TD_2(6, 2)$ to obtain a (6, 2)-HPMD($6^{13}8^1$). Now fill in the groups with an extra point. For $v = 69$ and 75, (6, 1)-HPMDs of types $8^7 12^1$ and $9^7 12^1$ are given below; take 2 copies of these designs and fill in the holes with 1 or 0 extra points.

For (6, 1)-HPMDs of types $g^7 12^1$ (with $g = 8$ or 9), we take the point set as $Z_{7g} \cup \{\infty_1, \infty_2, \ldots, \infty_{12}\}$. Multiply all the given blocks (except the first one for $9^7 12^1$) by $1, w, w^2$, where $w = 25$ for $8^7 12^1$ and 4 for $9^7 12^1$. Also, replace ∞_i $(i = 1, 4, 7, 10)$ by ∞_{i+1} and ∞_{i+2} when multiplying by w and w^2. Then develop the resulting base blocks (mod 56 or 63).

HPMD type	Base blocks
$8^7 12^1$	(0, 11, 15, 20, 44, ∞_1), (0, 2, 8, 40, 13, ∞_4), (0, 33, 36, 10, 6, ∞_7), (0, 23, 22, 47, 25, ∞_{10})
$9^7 12^1$	(51, 10, 15, 40, 60, 34), (0, 45, 12, 58, 48, ∞_1), (0, 9, 20, 4, 1, ∞_4), (0, 32, 10, 34, 26, ∞_7), (0, 3, 55, 26, 30, ∞_{10}).

4.3 Direct constructions for $\lambda = 3$

Now we give several direct constructions for $(v, 6, 3)$ PMDs. These are all obtained by developing the given base blocks (mod $v-1$, v, or $v-9$) as indicated. For the last one, $v = 54$, the given base blocks generate a (6, 3)-IPMD(54, 9); fill in the size 9 hole with a (9, 6, 3)-PMD from [36].

Lemma 4.8 *If* $v \in \{8, 10, 12, 14, 16, 18, 24, 30, 33, 34, 39, 54\}$*, a* $(v, 6, 3)$*-PMD exists.*

v	Point set	Base blocks
8	$Z_7 \cup \{\infty\}$	(0, 1, 2, 6, 4, ∞), (0, 2, 4, 5, 1, ∞), (0, 4, 1, 3, 2, ∞), (5, 1, 6, 4, 3, 2).
10	$Z_9 \cup \{\infty\}$	(0, 6, 8, 5, 3, ∞), (0, 5, 7, 2, 8, ∞), (0, 1, 2, 5, 7, ∞), (0, 3, 7, 8, 4, 2), (0, 5, 4, 8, 7, 6).
12	$Z_{11} \cup \{\infty\}$	(0, 3, 7, 4, 8, ∞), (0, 3, 2, 9, 8, ∞), (0, 5, 8, 9, 6, ∞), (0, 5, 1, 3, 9, 10), (0, 6, 4, 10, 9, 7), (0, 8, 10, 4, 2, 9).
14	$Z_{13} \cup \{\infty\}$	(0, 5, 7, 9, 12, ∞), (0, 4, 11, 12, 7, ∞), (0, 9, 4, 10, 7, ∞), (0, 1, 6, 12, 11, 2), (0, 2, 1, 4, 9, 3), (0, 7, 10, 5, 6, 2), (0, 6, 3, 2, 11, 9).
16	$Z_{15} \cup \{\infty\}$	(0, 9, 2, 8, 11, ∞), (0, 5, 11, 14, 1, ∞), (0, 12, 10, 4, 3, ∞), (0, 2, 10, 3, 12, 1), (0, 3, 1, 2, 6, 10), (0, 5, 4, 11, 8, 9), (0, 1, 12, 14, 9, 5), (0, 11, 3, 1, 13, 8).
18	$Z_{17} \cup \{\infty\}$	(0, 7, 9, 8, 12, ∞), (0, 6, 7, 4, 11, ∞), (0, 15, 8, 16, 11, ∞), (0, 1, 4, 9, 3, 11), (0, 5, 3, 6, 2, 16), (0, 5, 11, 2, 15, 10), (0, 13, 11, 3, 7, 1),(0, 2, 11, 15, 8, 3), (0, 9, 12, 14, 8, 7).
24	$Z_{23} \cup \{\infty\}$	(0, 17, 12, 8, 13, ∞), (0, 18, 9, 7, 4, ∞), (0, 7, 11, 4, 6, ∞), (0, 11, 8, 2, 14, 1), (0, 4, 1, 2, 9, 14), (0, 9, 1, 2, 12, 10), (0, 3, 22, 5, 16, 9), (0, 15, 2, 11, 4, 17), (0, 6, 4, 9, 17, 1), (0, 19, 20, 22, 2, 15), (0, 17, 16, 1, 13, 8), (0, 14, 18, 6, 8, 11).
30	$Z_{29} \cup \{\infty\}$	(0, 1, 28, 6, 9, ∞), (0, 21, 7, 3, 1, ∞), (0, 8, 28, 21, 19, ∞), (0, 2, 24, 4, 16, 5), (0, 4, 12, 26, 20, 1), (0, 5, 18, 24, 12, 9), (0, 11, 27, 17, 13, 28), (0, 2, 9, 22, 14, 17), (0, 4, 18, 5, 10, 7), (0, 6, 16, 10, 19, 18), (0, 1, 3, 10, 25, 16), (0, 3, 13, 25, 2, 11), (0, 4, 23, 28, 24, 6), (0, 8, 25, 15, 12, 11), (0, 14, 2, 26, 21, 8).
33	Z_{33}	(0, 7, 15, 26, 19, 11), (0, 2, 8, 22, 20, 14), (0, 9, 10, 28, 19, 18), (0, 10, 18, 1, 24, 16), (0, 28, 24, 3, 8, 12), (0, 29, 27, 9, 13, 15), (0, 1, 29, 3, 12, 6), (0, 17, 12, 9, 22, 10), (0, 30, 18, 29, 22, 19), (0, 2, 26, 15, 28, 32), (0, 18, 5, 19, 2, 25). (Multiply the last 5 blocks by 1 and -1)
34	$Z_{33} \cup \{\infty\}$	(0, 1, 24, 26, 32, 23), (0, 7, 18, 16, 4, 3), (0, 4, 17, 11, 23, 5), (0, 20, 13, 9, 28, ∞), (0, 8, 26, 29, 21, ∞), (0, 5, 14, 28, 17, ∞), (0, 1, 3, 12, 11, 9), (0, 1, 29, 3, 6, 21), (0, 8, 21, 6, 28, 25), (0, 6, 27, 14, 28, 4), (0, 29, 13, 18, 12, 22), (0, 16, 18, 2, 9, 19). (Multiply the last 5 blocks by 1 and -1)

v	Point set	Base blocks
39	Z_{39}	(1, 5, 16, 2, 22, 32), (0, 1, 13, 17, 5, 11), (0, 2, 29, 14, 6, 17), (0, 7, 16, 6, 35, 8), (0, 1, 19, 9, 14, 13), (0, 12, 7, 32, 13, 37), (0, 6, 27, 11, 24, 22). (Multiply the last 6 blocks by 1, 16 and 22)
54	$Z_{45} \cup \{\infty_1, \infty_2, \ldots, \infty_9\}$	(1, 8, 16, 38, 31, 23), (0, 23, 5, 29, 33, 31), (0, 2, 42, 3, 38, ∞_1), (0, 5, 19, 22, 44, ∞_2), (0, 1, 36, 21, 32, ∞_3), (0, 7, 2, 28, 37, ∞_4), (0, 18, 39, 27, 8, ∞_5), (0, 15, 31, 28, 26, ∞_6), (0, 43, 34, 1, 18, ∞_7), (0, 4, 23, 22, 16, ∞_8), (0, 23, 40, 11, 21, ∞_9). (Multiply the last 10 blocks by 1, 16 and 31)

□

5 Recusive constructions for $(v, 6, 2)$-PMDs

Now we can start constructing $(v, 6, 2)$-PMDs for larger v. We already have some preliminary upper bounds from [1]; these are stated in the next lemma.

Lemma 5.1 A $(v, 6, 1)$ *PMD exists in each of the following cases:* (1) $v \equiv 1$ (*mod* 6); (2) $v \equiv 0$ (*mod* 6) *and* $v > 198$; (3) $v \equiv 4$ (*mod* 6) *and* $v > 148$; (4) $v \equiv 3$ (*mod* 6) *and* $v > 657$.

When dealing with the larger cases, we shall often make use of the following lemma:

Lemma 5.2 *Suppose a TD*(10, *t*) *exists and* $v = 9t + u$ *where* $0 \le u \le 2t$. *Then there exists a* (6, 2)-*HPMD of type* t^9u^1. *If further,* $h = 0$ *or* 1 *and there exist a* $(t + h, 6, 2)$-*PMD and an* $(u + h, 6, 2)$-*PMD, then a* $(v, 6, 2)$-*PMD exists for* $v = 9t + u + h$.

Proof For the HPMD, start with a TD(10, t) and apply Construction 3.3, giving weight 1 to all points in the first 9 groups. In the last group, give the points weight 0, 1 or 2 so that the total weight is u. The required input designs are a (9, 6, 2)-PMD, a (10, 6, 2)-PMD and a (6, 2)-IPMD(11, 2); these are given in Lemmas 4.4 and 4.6. For the $(v, 6, 2)$-PMD, apply the fill in Construction 3.2 with $h = 0$ or 1. □

We first look at the 0 (mod 6) class.

Lemma 5.3 *If* $v \equiv 0$ (*mod* 6) *and* $v \ne 6$, *then a* $(v, 6, 2)$-*PMD exists.*

Proof For $v = 36, 42, 66, 78$ these designs can be obtained by taking two copies of a $(v, 6, 1)$-PMD [1], and the other values ≤ 66 (except $v = 6$) have been obtained directly in Lemmas 4.4 and 4.5. All other cases ≤ 198 (except 72, 84, 102, 150) can be handled by Lemma 5.2, using the values indicated in the following table.

t	h	v	t	h	v
9	0	90, 96	16	0	144, 156–174
11	1	108–120	17	1	162–186
13	0	126–138	19	0	180–204

For $v = 72$, 84 and 150, (6, 1)-HPMDs of types 3^8, 1^7 and 3^{10} exist [1]; apply Construction 3.4 to these, using a $TD_2(6, m)$ for $m = 3$, 12 or 5. This gives (6, 2)-HPMDs of types 9^8, 12^7 and 15^{10}; now fill in the holes with $(t, 6, 2)$-PMDs for $t = 9$, 12 or 15.

For $v = 102$, delete one block in a TD(10, 11) and use a deleted point to redefine groups, giving a {9, 10}-GDD of type $9^{10}10^1$. Applying Construction 3.3, giving all points weight 1, gives a (6, 2)-HPMD of the same type. Now fill in the groups with 2 extra points using a (6, 2)-IPMD(11, 2) and a (12, 6, 2)-PMD. This completes the proof. □

Lemma 5.4 *If* $v \equiv 3$ *or* 4 (*mod* 6) *and* $v \geq 9$, *then a* $(v, 6, 2)$-*PMD exists.*

Proof For $v = 28, 40, 46, 147$, $(v, 6, 1)$-PMDs exist [1]. Direct constructions are given for other such v, $9 \leq v \leq 46$ in Lemma 4.4 and for $v = 52, 58, 69, 75, 87$, in Lemma 4.7. For $v = 51$, a (6, 1)-IPMD(51, 10) was obtained in Lemma 4.6; take two copies of this design and fill in the hole with a (10, 6, 2)-PMD. For $v = 64$ and 70, apply Construction 3.4, noting that a $(v, 7, 2)$ BIBD exists [7]. For 57 and 63, we have a $(v, \{7, 9\}, 1)$-PBD, either from a TD(7, 9) or by adding a point to the groups of a TD(7, 8); again apply Construction 3.4, noting (7, 6, 2) and (9, 6, 2)-PMDs exist. For $v = 76$, a (6, 1)-IPMD(76, 15) is obtained directly from a V(4, 15) vector in [23]; take two copies of this and fill in the hole with a (15, 6, 2)-PMD. For 105, and 148, apply Construction 3.5, inflating a (6, 1)-HPMD of type 1^7 with a $TD_2(6, m)$ for $m = 15$ or 21, then fill in the holes with 0 or 1 extra points.

For 123, start with a TD(14, 13), then delete all points in the last group and all points from groups 10 to 13 except those in a specific block B. Using the blocks containing the deleted point in B from group 14 to redefine groups gives a {9, 10, 13}-GDD of type $9^{12}13^1$. Applying Construction 3.3 giving all points in this GDD weight one produces a (6, 2)-HPMD of the same type. Now fill in the holes with two extra points, using a (6, 2)-IPMD(11, 2) and a (15, 6, 2)-PMD.

For 213, truncating one group of a TD(13, 17) to size 8 gives a {12, 13}-GDD of type $17^{12}8^1$. Apply Construction 3.3, giving all points in this GDD weight 1, then fill in the holes with 1 extra point, using (18, 6, 2) and (9, 6, 2) PMDs.

All the remaining cases with $81 \leq v \leq 657$ can be handled by Lemma 5.2. Suitable values of t and h are given in the next table.

t	h	v	t	h	v
9	0	81–82, 88–99	27	0	250–297
11	1	100, 106–118	31	0	286–340
13	0	124–142	37	0	340–406
16	0	153–172	43	0	394–472
17	1	160–184	49	0	448–538
19	0	178–208	53	1	484–580
23	1	214–250	61	0	554–670

□

Summarizing the results of this section, we now have

Theorem 5.5 *If* $v \equiv 0, 1, 3, 4$ (*mod* 6) *then a* $(v, 6, 2)$-*PMD exists, except for* $v = 6$.

6 $(v, 6, 3)$-PMDs

We now look at constructions for $(v, 6, 3)$ PMDs. Here we need an extra construction lemma:

Lemma 6.1 *Suppose* $u = (h_1 + h_2 + h_3)/3$ *is an integer, and either* (1) *there exists a* (6, 1)-*IPMD*$(t + h_i, h_i)$ *for* $i = 1, 2, 3$ *or* (2) $h_1 = h_2 = 0$ *and there exist a* $(t, 6, 2)$-*PMD plus a* (6, 1)-*IPMD*$(t + h_3, h_3)$. *Then a* (6, 3)-*IPMD*$(t + u, u)$ *exists. If further, either* $u \in \{0, 1\}$ *or a* $(u, 6, 3)$-*PMD exists, then a* $(t + u, 6, 3)$-*PMD also exists.*

Proof Let T be a set of size t. For $i = 1, 2, 3$, construct either a (6, 1)-IPMD$(t + h_i, h_i)$ for $i = 1, 2, 3$ or a $(t, 6, 2)$-PMD and a (6, 1)-IPMD$(t + h_3, h_3)$. Each of these designs should be on T plus h_i extra points, with the size h_i hole being on the extra points. Now replace each extra point by one point from $U = \{\infty_1, \infty_2, \ldots, \infty_u\}$ in such a way that each point from U replaces exactly 3 extra points. Finally form a $(u, 6, 3)$-PMD on U.

We now consider $(v, 6, 3)$-PMDs for $v \le 62$:

Lemma 6.2 *If* $7 \le v \le 62$*, then a* $(v, 6, 3)$*-PMD exists.*

Proof Yin in [36] constructed all cases with v an odd prime power and $v = 8, 14$; solutions for these last two designs are also given in Lemma 4.8. Yin also obtained 15, 21, 35 by Construction 3.4, using a (7, 6, 1) PMD and $(v, 7, 3)$ BIBDs from [27]. For $v = 28, 36, 40, 42, 46, 55$, a $(v, 6, 1)$ PMD exists [1], while $v = 52, 58$ can be handled by taking 3 copies of (6, 1) HPMDs of types $1^{43}9^1$, $7^7 9^1$ from [1], and filling the holes with (7, 6, 3) or (9, 6, 3)-PMDs. The cases $v = 10, 12, 16, 18, 24, 30, 33, 39, 54$ were handled in Lemma 4.8, while $v = 20, 26, 32, 38, 44, 62$ were obtained in Theorem 10.2 of [1], using Lemma 6.1 with $t = v - 1$, $h_1 = h_2 = 0$ and $h_3 = 3$. This leaves just $v = 22, 45, 48, 50, 56, 60$ to be handled. For $v = 50$ and 56, inflate a (6, 3)-HPMD of type 1^7 with a TD(6, m) for $m = 7$ or 8, and then fill in the holes with 1 or 0 extra points, using an (8, 6, 3)-PMD. For the others we apply Lemma 6.1 with the following parameters:

v	t	h_1, h_2, h_3	v	t	h_1, h_2, h_3
22	21	0, 0, 3	48	41	10, 10, 1
45	37	9, 9, 6	60	49	12, 12, 9

The required input designs here are a (21, 6, 2)-PMD (given in Lemma 4.4) and a (6, 1)-IPMD(t, h) for $(t, h) = (24, 3), (43, 6), (46, 9), (42, 1), (51, 10), (58, 9)$ and $(61, 12)$. The case $(t, h) = (46, 9)$ is obtainable from a V(4, 9) vector from [23], while $(t, h) = (51, 10)$ is given in Lemma 4.6. Also, (42, 1) and (58, 9) can be obtained by filling in holes of a (6, 1)-IPMD(42, 7) and a (6, 1)-HPMD of type $7^7 9^1$; these designs are in Lemmas A.1 and A.8 of [1]. For (24, 3) and (61,12), see Lemmas A.3 and A.7 in [1]. □

In [36], a construction was given for $(v, 6, 3)$-PMDs for $v > 62$. For the sake of completeness and for the benefit of the reader, we provide a simplified version of this proof in the following theorem.

Theorem 6.3 *A* $(v, 6, 3)$*-PMD exists for all integers* $v \ge 6$*, except for* $v = 6$.

Proof In view of Lemma 6.2, it is sufficient to prove the result for all $v \ge 63$. For all such v, we mention a construction for a $(v, K, 1)$-PBD with $K = \{7, 8, 9, \ldots, 19\}$. Since $(t, 6, 3)$-PMDs exist for $7 \le t \le 19$, the result will then follow from Construction 3.4.

For $63 \le v \le 92$, or $v \ge 343$, a $(v, \{7, 8, 9\}, 1)$-PBD is known to exist [30]. For $93 \le v \le 361$, one can truncate up to $q - 7$ groups in a TD(q, q) for $q = 11, 13, 16, 17$, or 19 to sizes ≥ 7. □

7 Summary

Theorems 4.2, 4.3, 5.5 and 6.3 have established that the necessary conditions for a $(v, 6, \lambda)$-PMD are sufficient (1) for $\lambda \in \{2, 3\}$, $v \ne 6$ and (2) for $\lambda \in \{4, 6\}$, $v = 6$. Further, all feasible $(v, 6, \lambda)$-PMDs with larger λ can easily be obtained from these by Lemma 1.8. Consequently, we have achieved our goal of establishing Theorem 1.7, which we restate as follows:

Theorem 7.1 *Necessary conditions for existence of a* $(v, 6, \lambda)$*-PMD are* (1) $v \ge 6$*, and* (2) $v \equiv 0$ *or* 1 (*mod* 3) *if* $\lambda \not\equiv 0$ (*mod* 3)*. For* $\lambda > 1$*, these are sufficient, except there is no* $(6, 6, \lambda)$*-PMD for* $\lambda = 2$ *or* λ *odd.*

We conclude this paper by mentioning the existence problem for almost-resolvable $(v, k, 1)$-PMDs. (A PMD is called *almost resolvable* if its blocks can be partitioned into near parallel classes where a near parallel class is a set of blocks missing one point, and containing all other points once.) For such a design to exist, we require $v \equiv 1 \pmod{k}$. For $k = 3, 4, 5$, these designs are known to exist for all such v, except for $k = 5, v = 6$, and possibly for $k = 5, v = 26$ [4, 9, 16, 20, 29]. For $k = 6$, these designs are known to exist whenever a $(v, Q_{1(6)}, 1)$-PBD exists [10]. Further, from [13], such a PBD exists for all $v \equiv 1$ (mod 6), except for $v = 55$ and possibly 21 other cases. For eight of these ($v = 55, 115, 145, 445, 685, 745, 799, 805$), an almost-resolvable $(v, 6, 1)$-PMD can be found in [6]. The next theorem summarizes the remaining open cases.

Theorem 7.2 *The necessary condition for the existence of an almost resolvable* $(v, 6, 1)$-*PMD, namely* $v \equiv 1$ $(mod\ 6)$*, is sufficient except possibly for the* 14 *cases* $v = 205, 235, 265, 319, 355, 391, 415, 451, 493, 649, 667, 697, 781, 1315$.

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