Designs, Codes and Cryptography, 26, 169–196, 2002

The Existence of Kirkman Squares—Doubly Resolvable $(v, 3, 1)$-*BIBD*s

CHARLES J. COLBOURN Charles.Colbourn@asu.edu
Computer Science and Engineering, Arizona State University, Tempe AZ 85287-5406, U.S.A.

E. R. LAMKEN lamken@its.caltech.edu
Mathematics, 253-37, California Institute of Technology, California, CA 91125, U.S.A.

ALAN C. H. LING aling@emba.uvm.edu
Computer Science, University of Vermont, Burlington, VT 05405, U.S.A.

W. H. MILLS
CCR-IDA, Thanet Road, Princeton, NJ 08540, U.S.A.

Communicated by: D. R. Stinson, G. H. J. van Rees

Abstract. A Kirkman square with index λ, latinicity μ, block size k, and v points, $KS_k(v; \mu, \lambda)$, is a $t \times t$ array $(t = \lambda(v-1)/\mu(k-1))$ defined on a v-set V such that (1) every point of V is contained in precisely μ cells of each row and column, (2) each cell of the array is either empty or contains a k-subset of V, and (3) the collection of blocks obtained from the non-empty cells of the array is a (v, k, λ)-*BIBD*. For $\mu = 1$, the existence of a $KS_k(v; \mu, \lambda)$ is equivalent to the existence of a doubly resolvable (v, k, λ)-*BIBD*. The spectrum of $KS_2(v; 1, 1)$ or Room squares was completed by Mullin and Wallis in 1975. In this paper, we determine the spectrum for a second class of doubly resolvable designs with $\lambda = 1$. We show that there exist $KS_3(v; 1, 1)$ for $v \equiv 3 \pmod 6$, $v = 3$ and $v \geq 27$ with at most 23 possible exceptions for v.

Keywords: Steiner triple system, resolvable, doubly resolvable, Kirkman square, Kirkman triple system

AMS Subject Classification: 05B07

1. Introduction

A Kirkman square with index λ, latinicity μ, block size k, and v points, $KS_k(v; \mu, \lambda)$, is a $t \times t$ array $(t = \lambda(v-1)/\mu(k-1))$ defined on a v-set V such that

(1) every point of V is contained in precisely μ cells of each row and column,

(2) each cell of the array is either empty or contains a k-subset of V, and

(3) the collection of blocks obtained from the non-empty cells of the array is a (v, k, λ)-*BIBD*.

The existence of a $KS_2(v; \mu, \lambda)$ has been completely settled [24,30,31].

The existence of a $KS_k(v; \mu, \lambda)$ is equivalent to the existence of a μ-resolvable (v, k, λ)-*BIBD* with a pair of orthogonal μ-resolutions [30]. In particular, the existence of a

$KS_k(v; 1, \lambda)$ is equivalent to the existence of a doubly resolvable (v, k, λ)-*BIBD*. In this paper, we are interested in the existence of $KS_k(v; 1, \lambda)$ or $DR(v, k, \lambda)$-*BIBD*s.

A balanced incomplete block design (*BIBD*) D is a collection B of subsets (blocks) taken from a finite set V of v elements with the properties:

(1) Every pair of distinct elements of V is contained in precisely λ blocks of B.

(2) Every block contains exactly k elements.

We denote such a design as a (v, k, λ)-*BIBD*. The necessary conditions for the existence of a (v, k, λ)-*BIBD* are

$$\lambda(v-1) \equiv 0 \pmod{(k-1)} \text{ and } \lambda v(v-1) \equiv 0 \pmod{k(k-1)}. \qquad (*)$$

A (v, k, λ)-*BIBD* D is said to be resolvable (and denoted by (v, k, λ)-*RBIBD*) if the blocks of D can be partitioned into classes $R_1, R_2, \ldots, R_r$ (resolution classes) where $r = \frac{\lambda(v-1)}{(k-1)}$ such that each element of D is contained in precisely one block of each class. The classes $R_1, R_2, \ldots, R_r$ form a resolution of D. The necessary conditions for the existence of a (v, k, λ)-*RBIBD* are $(*)$ and $v \equiv 0 \pmod{k}$.

A (v, k, λ)-*BIBD* is said to be doubly resolvable if there exist two resolutions R and R' of the blocks such that $|R_i \cap R'_j| \leq 1$ for all $R_i \in R$ and $R'_j \in R'$. (It should be noted that the blocks of the design are considered as being labeled so that if a subset of the element set occurs as a block more than once the blocks are treated as distinct.) The resolutions R and R' are called orthogonal resolutions of the design. A doubly resolvable (v, k, λ)-*BIBD* is denoted by $DR(v, k, \lambda)$-*BIBD*.

We can use a pair of orthogonal resolutions of a $DR(v, k, \lambda)$-*BIBD* to construct a $KS_k(v; 1, \lambda)$. We index the rows and columns of an $r \times r$ array with a pair of orthogonal resolutions, R and R'. In the cell labeled (R_i, R'_j), we place $R_i \cap R'_j$ for all $R_i \in R$ and $R'_j \in R'$. If $R_i \cap R'_j = \emptyset$, the cell is left empty. It is easy to verify that this array is a $KS_k(v; 1, \lambda)$. Similarly, it is easy to see that a $KS_k(v; 1, \lambda)$ displays a pair of orthogonal resolutions of a $DR(v, k, \lambda)$-*BIBD*.

To illustrate these definitions, a $KS_3(27; 1, 1)$ is displayed in Figure 1. The rows of the $KS_3(27; 1, 1)$ form the resolution classes of one resolution of the $DR(27, 3, 1)$-*BIBD* and the columns contain the resolution classes of an orthogonal resolution. This design has a

$6_0 2_1 4_2$		$5_0 6_1 7_2$		$7_1 2_2 \infty_0$		$7_0 5_1 \infty_2$	$4_0 1_2 \infty_1$	$2_0 4_1 3_2$	$0_2 5_2 6_2$	$0_1 1_1 3_1$		$0_0 1_0 3_0$
$5_0 2_2 \infty_1$	$7_0 3_1 5_2$		$6_0 7_1 0_2$		$0_1 3_2 \infty_0$		$0_0 6_1 \infty_2$	$3_0 5_1 4_2$	$1_0 2_0 4_0$	$1_2 6_2 7_2$	$1_1 2_1 4_1$	
$1_0 7_1 \infty_2$	$6_0 3_2 \infty_1$	$0_0 4_1 6_2$		$7_0 0_1 1_2$		$1_1 4_2 \infty_0$		$4_0 6_1 5_2$		$2_0 3_0 5_0$	$2_2 7_2 0_2$	$2_1 3_1 5_1$
	$2_0 0_1 \infty_2$	$7_0 4_2 \infty_1$	$1_0 5_1 7_2$		$0_0 1_1 2_2$		$2_1 5_2 \infty_0$	$5_0 7_1 6_2$	$3_1 4_1 6_1$		$3_0 4_0 6_0$	$3_2 0_2 1_2$
$3_1 6_2 \infty_0$		$3_0 1_1 \infty_2$	$0_0 5_2 \infty_1$	$2_0 6_1 0_2$		$1_0 2_1 3_2$		$6_0 0_1 7_2$	$4_2 1_2 2_2$	$4_1 5_1 7_1$		$4_0 5_0 7_0$
	$4_1 7_2 \infty_0$		$4_0 2_1 \infty_2$	$1_0 6_2 \infty_1$	$3_0 7_1 1_2$		$2_0 3_1 4_2$	$7_0 1_1 0_2$	$5_0 6_0 0_0$	$5_2 2_2 3_2$	$5_1 6_1 0_1$	
$3_0 4_1 5_2$		$5_1 0_2 \infty_0$		$5_0 3_1 \infty_2$	$2_0 7_2 \infty_1$	$4_0 0_1 2_2$		$0_0 2_1 1_2$		$6_0 7_0 1_0$	$6_2 3_2 4_2$	$6_1 7_1 1_1$
	$4_0 5_1 6_2$		$6_1 1_2 \infty_0$		$6_0 4_1 \infty_2$	$3_0 0_2 \infty_1$	$5_0 1_1 3_2$	$1_0 3_1 2_2$	$7_1 0_1 2_1$		$7_0 0_0 2_0$	$7_2 4_2 5_2$
$0_0 0_1 0_2$	$1_0 1_1 1_2$	$2_0 2_1 2_2$	$3_0 3_1 3_2$	$4_0 4_1 4_2$	$5_0 5_1 5_2$	$6_0 6_1 6_2$	$7_0 7_1 7_2$	$\infty_0 \infty_1 \infty_2$				
	$5_0 2_1 0_2$	$4_0 7_1 3_2$	$2_0 1_1 6_2$		$1_0 6_1 4_2$	$0_0 3_1 7_2$	$6_0 5_1 2_2$		$3_0 7_0 \infty_0$		$1_2 5_2 \infty_2$	$0_1 4_1 \infty_1$
$7_0 6_1 3_2$		$6_0 3_1 1_2$	$5_0 0_1 4_2$	$3_0 2_1 7_2$		$2_0 7_1 5_2$	$1_0 4_1 0_2$		$1_1 5_1 \infty_1$	$4_0 0_0 \infty_0$		$2_2 6_2 \infty_2$
$2_0 5_1 1_2$	$0_0 7_1 4_2$		$7_0 4_1 2_2$	$6_0 1_1 5_2$	$4_0 3_1 0_2$		$3_0 0_1 6_2$		$3_2 7_2 \infty_2$	$2_1 6_1 \infty_1$	$5_0 1_0 \infty_0$	
$4_0 1_1 7_2$	$3_0 6_1 2_2$	$1_0 0_1 5_2$		$0_0 5_1 3_2$	$7_0 2_1 6_2$	$5_0 4_1 1_2$				$4_2 0_2 \infty_2$	$3_1 7_1 \infty_1$	$6_0 2_0 \infty_0$

Figure 1. A $KS_3(27; 1, 1)$.

particularly nice structure; its automorphism group partitions the rows and the columns into orbits of lengths 8, 4, and 1. (The double lines in the figure make the orbit structure clear.)

In recent years, a great deal of progress has been made on the existence of resolvable balanced incomplete block designs. Necessary and sufficient conditions for the existence of (v, k, λ)-*RBIBD*s are known for several (k, λ) pairs, and the necessary conditions are also known to be sufficient with a finite number of possible exceptions for several other (k, λ) pairs; see [3,16]. In addition, the asymptotic existence was established in 1984 [34].

In contrast, very little is known about the existence of doubly resolvable balanced incomplete block designs or Kirkman squares. Although infinite classes of $DR(v, k, \lambda)$-*BIBD*s have been constructed for certain values of k, $k \geq 3$, the spectrum has been determined for only two cases. A $DR(v, 2, 1)$-*BIBD* or a $KS_2(v; 1, 1)$ is also known as a Room square of side $v - 1$. The first example of a Room square, a $RS(7)$, was constructed by Kirkman in 1850 [19]. The spectrum of Room squares was completed over 120 years later by Mullin and Wallis [37].

THEOREM 1.1 [37]. *There exists a* $KS_2(v; 1, 1)$ *or a* $RS(v - 1)$ *for* v *a positive integer,* $v \equiv 0 \pmod 2$ *and* $v \neq 4$ *or* 6.

An extensive bibliography is available on Room squares and Room squares with special properties, see for example [37] and the more recent survey [12].

Quite a lot of work was done after the Room square problem was completed in 1975 to try to determine the spectrum for a class of doubly resolvable (v, k, λ)-*BIBD*s with $k \geq 3$ [5,9,10,11,13,14,15,32,35,36,38,39,40,43,42,44,45]. Although considerable effort was put into the case $k = 3$ and $\lambda = 1$, the case $k = 3$ and $\lambda = 2$ turned out to be the natural analogue of the Room square case. The spectrum of a second class of doubly resolvable designs was finally established in 1995 by using the connection between partitioned generalized balanced tournament designs and Kirkman squares [26,28].

THEOREM 1.2 [26]. *Let* v *be a positive integer,* $v \equiv 0 \pmod 3$*,* $v \neq 6, 9$. *There exists a* $KS_3(v; 1, 2)$ *except possibly for* $v \in \{72, 78, 90, 114, 117, 126\}$. *Furthermore, there do not exist* $KS_3(v; 1, 2)$ *for* $v = 6$ *and* $v = 9$.

For completeness, we note that the spectrum of another class of Kirkman squares with block size 3, $KS_3(v; 2, 4)$s, has also been determined [27]. In addition, the asymptotic existence of $DR(v, k, 1)$-*BIBD*s was recently shown in [25].

In this paper, we return to the case of $k = 3$ and $\lambda = 1$, the elusive $KS_3(v; 1, 1)$ or $DR(v, 3, 1)$-*BIBD*. A necessary condition for the existence of a $KS_3(v; 1, 1)$ is $v \equiv 3 \pmod 6$. We determine the spectrum of $KS_3(v; 1, 1)$s with at most 23 possible exceptions for v.

In the next section, we describe some of the previous work on the existence of $KS_3(v; 1, 1)$s. One of the major difficulties in determining the spectrum of $KS_3(v; 1, 1)$s is the construction of small examples, particularly when $v \equiv 9 \pmod{12}$. Section 3 contains several new direct constructions for $KS_3(v; 1, 1)$s. Our main recursive construction uses frames; Section 4 contains definitions and existence results for these frames. Finally, in Section 5 we construct $KS_3(v; 1, 1)$s for $v \equiv 3 \pmod 6$ with at most 23 possible exceptions for v.

2. Previous Results

In this section, we collect the earlier existence results for $KS_3(v; 1, 1)$s. It is important to note that only 5 nontrivial $KS_3(v; 1, 1)$ had been constructed for $v < 100$.

The array [123] is a $KS_3(3; 1, 1)$. The nonexistence of the next case, a $KS_3(9; 1, 1)$, is also easy to establish since there is only one $(9, 3, 1)$-*RBIBD* (constructed from the affine plane) and it is not doubly resolvable [36]. The nonexistence of a $KS_3(15; 1, 1)$ is a bit harder to show. Mathon and Vanstone used the equivalence between the existence of equidistant permutation arrays and the existence of doubly resolvable designs to show that a $KS_3(15; 1, 1)$ cannot exist [36]. Thus, the smallest open case is $v = 21$.

The first example of a $KS_3(v; 1, 1)$ was constructed in 1979 for $v = 27$. It is the smallest case of an infinite family constructed from finite geometries and sub-designs in geometries. The $KS_3(27; 1, 1)$ constructed from the affine plane has the same nice structure as the design in Figure 1; its automorphism group partitions the rows and columns into orbits of lengths 8, 4, and 1.

THEOREM 2.1 [36,15]. *For q a prime or prime power, there exists a $KS_q(q^3; 1, 1)$.*

This result was generalized in 1987 using orthogonal resolutions of the lines in affine geometries, $AG(n, q)$. It produces another small design for $v < 100$, a $KS_3(81; 1, 1)$.

THEOREM 2.2 [14]. *For q a prime or prime power and n any integer, $n \geq 3$, there exists a $KS_q(q^n; 1, 1)$.*

Orthogonal line packings in projective geometries can also be used to construct an infinite class of Kirkman squares [5]. The smallest nontrivial example of this construction is a $KS_3(63; 1, 1)$.

THEOREM 2.3 [5]. *For all $m \neq 1 \pmod 3$ there exists a $KS_3(2^{2m+2} - 1; 1, 1)$.*

The remaining two designs previously known for $v < 100$ were constructed using the idea of embedding strong starters for Room squares in triple systems [40]. Both required computer searches.

LEMMA 2.4 [40,42]. *There exist $KS_3(v; 1, 1)$ for $v = 39$ and $v = 51$.*

One of the first recursive constructions for Kirkman squares is the pairwise balanced design (PBD) construction [36]. A pairwise balanced design, $PBD(v; K)$ is a set B of subsets taken from a finite set V of v elements with the properties that every pair of distinct elements from V is contained in precisely one block of B and $|b| \in K$ for all $b \in B$.

THEOREM 2.5 [36]. *If there exists a $PBD(v; K)$ and if for each $k \in K$ there exists a $KS_3(2k + 1; 1, 1)$, then there exists a $KS_3(2v + 1; 1, 1)$.*

In the language of PBD closure, Theorem 2.5 says that this class of Kirkman squares is PBD-closed for the set of replication numbers. (See [46] for definitions and results on

PBD closure.) This result was used together with the existence of two small $KS_3(v; 1, 1)$s, $v = 39$ and $v = 81$, to prove the asymptotic existence of $KS_3(v; 1, 1)$s in [40].

THEOREM 2.6 [40]. *There exists a constant v_0 such that for all $v > v_0$ and $v \equiv 3 \pmod 6$, there exists a $KS_3(v; 1, 1)$.*

3. New Direct Constructions for $KS_3(v; 1, 1)$

Our first direct construction uses starters and adders. A starter S for a $KS_3(6n + 3; 1, 1)$ defined on $(\mathbb{Z}_{3n+1} \times \{0, 1\}) \cup \{\infty\}$ is a partition of $(\mathbb{Z}_{3n+1} \times \{0, 1\}) \cup \{\infty\}$ into $2n + 1$ triples, $T_1, T_2, \ldots, T_{2n+1}$, such that every nonzero element of $\mathbb{Z}_{3n+1}$ occurs exactly once as a pure (j, j) difference for $j = 0, 1$ and every element of $\mathbb{Z}_{3n+1}$ occurs exactly once as a mixed (i, j) difference for $i \neq j$, $i, j = 0, 1$. (See [4,7] for definitions of pure and mixed differences.) Let $A(S) = \{a_1, a_2, \ldots, a_{2n+1}\}$ where $a_i \in \mathbb{Z}_{3n+1}$ and no element of $\mathbb{Z}_{3n+1}$ occurs more than once in $A(S)$. $A(S)$ is an adder for S if $S + A(S) = \{T_i + a_i \mid i = 1, 2, \ldots, 2n + 1\} = (\mathbb{Z}_{3n+1} \times \{0, 1\}) \cup \{\infty\}$. (If $T_i = \{x_i, y_i, z_i\}$, then $T_i + a_i = \{x_i + a_i, y_i + a_i, z_i + a_i\}$ where all arithmetic is done modulo $3n + 1$.)

THEOREM 3.1. *If there exists a starter S for a $KS_3(6n + 3; 1, 1)$ on $(\mathbb{Z}_{3n+1} \times \{0, 1\}) \cup \{\infty\}$ and a corresponding adder $A(S)$, then there exists a $KS_3(6n + 3; 1, 1)$.*

Proof. Let K be a $(3n + 1) \times (3n + 1)$ array. Label the columns of K by $0, 3n, 3n - 1, \ldots, 2, 1$, and label the rows of K by $0, 1, 2, \ldots, 3n$. In row 0 and column a_i, place the triple T_i of S for $i = 1, 2, \ldots, 2n + 1$. The array is generated cyclically from this row. In row j and column $a_i - j$, place the triple $T_i + j$ for $j = 1, 2, \ldots, 3n$. (All arithmetic is done modulo $3n + 1$ and $\infty + a = \infty$ for all $a \in \mathbb{Z}_{3n+1}$.) Every element of $(\mathbb{Z}_{3n+1} \times \{0, 1\}) \cup \{\infty\}$ occurs precisely once in each row and column of the resulting array; this is guaranteed by the properties of the starter and the corresponding adder. It is straightforward to check that the collection of triples in the non-empty cells of the array is a $(6n + 3, 3, 1)$-*BIBD*. ■

This starter-adder construction is a generalization of the construction embedding strong starters for Room squares in triple systems [40]. We use Theorem 3.1 to construct several Kirkman squares.

LEMMA 3.2. *There exists a $KS_3(2p + 1; 1, 1)$ for*

$$p \in \{43, 67, 79, 103, 127, 139, 151, 199, 223\}.$$

Proof. Let $V = (\mathbb{Z}_p \times \{0, 1\}) \cup \{\infty\}$. Each design contains the starter block $\{\infty, (0, 0), (0, 1)\}$ with the corresponding adder 0. For each of the designs, four additional starter blocks and their adders are shown in Table 1. The remaining starter blocks and the corresponding adders are obtained by multiplying by the elements of the subgroup of order $\frac{p-1}{6}$ generated by g. ■

LEMMA 3.3. *There exists a $KS_3(2p + 1; 1, 1)$ for $p \in \{37, 61, 109, 157\}$.*

Table 1. Constructions for $2p + 1$ with four blocks.

p	g	Starter	Adder	Starter	Adder
43	41	{(1, 0), (2, 0), (14, 0)}	1	{(3, 0), (1, 1), (2, 1)}	3
		{(6, 0), (9, 1), (23, 1)}	22	{(7, 0), (30, 1), (33, 1)}	28
67	64	{(1, 0), (2, 0), (8, 0)}	1	{(4, 0), (1, 1), (2, 1)}	2
		{(6, 0), (5, 1), (17, 1)}	5	{(7, 0), (11, 1), (55, 1)}	19
79	18	{(1, 0), (2, 0), (5, 0)}	1	{(3, 0), (1, 1), (2, 1)}	2
		{(6, 0), (3, 1), (7, 1)}	9	{(12, 0), (17, 1), (19, 1)}	68
103	72	{(1, 0), (2, 0), (7, 0)}	1	{(3, 0), (1, 1), (2, 1)}	2
		{(5, 0), (6, 1), (12, 1)}	45	{(6, 0), (17, 1), (22, 1)}	42
127	94	{(1, 0), (3, 0), (10, 0)}	3	{(9, 0), (1, 1), (3, 1)}	2
		{(13, 0), (15, 1), (22, 1)}	20	{(29, 0), (85, 1), (119, 1)}	81
139	64	{(1, 0), (2, 0), (4, 0)}	1	{(3, 0), (1, 1), (2, 1)}	5
		{(8, 0), (3, 1), (5, 1)}	12	{(9, 0), (10, 1), (28, 1)}	56
151	148	{(1, 0), (2, 0), (6, 0)}	1	{(3, 0), (1, 1), (2, 1)}	6
		{(4, 0), (6, 1), (12, 1)}	10	{(12, 0), (22, 1), (26, 1)}	35
199	132	{(1, 0), (2, 0), (11, 0)}	1	{(3, 0), (1, 1), (2, 1)}	2
		{(4, 0), (6, 1), (9, 1)}	34	{(6, 0), (55, 1), (152, 1)}	155
223	60	{(1, 0), (3, 0), (13, 0)}	2	{(5, 0), (1, 1), (6, 1)}	13
		{(9, 0), (18, 1), (19, 1)}	29	{(19, 0), (22, 1), (108, 1)}	85

Table 2. Constructions for $2p + 1$ with eight blocks.

p	g	Starter	Adder	Starter	Adder
37	10	{(1, 0), (2, 0), (4, 0)}	1	{(5, 0), (11, 0), (1, 1)}	3
		{(6, 0), (18, 0), (4, 1)}	6	{(7, 0), (12, 0), (16, 1)}	15
		{(14, 0), (8, 1), (35, 1)}	11	{(17, 0), (19, 1), (36, 1)}	9
		{(21, 0), (2, 1), (31, 1)}	13	{(7, 1), (25, 1), (32, 1)}	29
61	9	{(1, 0), (2, 0), (6, 0)}	2	{(3, 0), (5, 0), (1, 1)}	4
		{(4, 0), (12, 0), (7, 1)}	1	{(13, 0), (23, 0), (10, 1)}	8
		{(8, 0), (12, 1), (15, 1)}	15	{(10, 0), (33, 1), (45, 1)}	32
		{(26, 0), (37, 1), (52, 1)}	6	{(4, 1), (6, 1), (30, 1)}	27
109	38	{(1, 0), (2, 0), (4, 0)}	2	{(3, 0), (9, 0), (1, 1)}	8
		{(6, 0), (24, 0), (2, 1)}	3	{(8, 0), (29, 0), (5, 1)}	11
		{(11, 0), (10, 1), (18, 1)}	1	{(18, 0), (4, 1), (29, 1)}	13
		{(31, 0), (40, 1), (68, 1)}	6	{(9, 1), (59, 1), (60, 1)}	43
157	130	{(1, 0), (2, 0), (4, 0)}	1	{(3, 0), (8, 0), (1, 1)}	3
		{(5, 0), (18, 0), (2, 1)}	8	{(6, 0), (66, 0), (9, 1)}	24
		{(9, 0), (3, 1), (4, 1)}	37	{(11, 0), (7, 1), (24, 1)}	43
		{(13, 0), (31, 1), (63, 1)}	81	{(6, 1), (43, 1), (113, 1)}	64

Proof. Let $V = (\mathbb{Z}_p \times \{0, 1\}) \cup \{\infty\}$. Each of the designs contains the starter block $\{\infty, (0, 0), (0, 1)\}$ with the corresponding adder 0. For each of the designs, eight additional starter blocks and their adders are shown in Table 2. The remaining starter blocks and the corresponding adders are obtained by multiplying by the elements of the subgroup of order $\frac{p-1}{12}$ generated by g. ■

Next we construct two designs in the class $v \equiv 9 \pmod{12}$. In each case, we searched for starters and adders for $KS_3(3n+3; 1, 1)$ with automorphism groups which partitioned the rows and columns into orbits of lengths n, $n/2$, and 1. (This is the same nice structure as the $KS_3(27; 1, 1)$ in Figure 1.)

Let $n = 2m$. A starter S for a $KS_3(3n+3; 1, 1)$ defined on $V = (\mathbb{Z}_n \times \{0, 1, 2\}) \cup \{\infty, \alpha, \beta\}$ consists of six sets of triples, $S_1, S_2, \dots, S_6$ with the following structure.

(1) $|S_1| = m + 1$.
$S_1 = \{\{\infty, (r_1, 0), (r_2, 1)\}, \{\beta, (s_1, 1), (s_2, 2)\}, \{\alpha, (t_1, 0), (t_2, 2)\}\} \cup \{\{x_i, y_i, z_i\} \mid i = 1, 2, \dots, m-2\}$ where $r_i, s_i, t_i \in \mathbb{Z}_n$ and $x_i, y_i, z_i \in \mathbb{Z}_n \times \{0, 1, 2\}$.
$S_1 = \{T_1^1, T_2^1, \dots, T_{m+1}^1\}$

(2) $|S_2| = 1$.
$S_2 = \{(u_0, 0), (u_1, 1), (u_2, 2)\}$, $u_i \in \mathbb{Z}_n$.

(3) $|S_3| = m - 1$.
$S_3 = \{\{x_i, y_i, z_i\} \mid i = m-1, \dots, n-3\}$ where $x_i, y_i, z_i \in \mathbb{Z}_n \times \{0, 1, 2\}$.
$S_3 = \{T_1^3, T_2^3, \dots, T_{m-1}^3\}$.

(4) $|S_4| = 1$.
$S_4 = \{(v_0, 0), (v_1, 1), (v_2, 2)\}$, $v_i \in \mathbb{Z}_n$.

(5) $|S_5| = m - 1$.
$S_3 = \{\{x_i', y_i', z_i'\} \mid i = 1, \dots, m-1\}$ where $x_i', y_i', z_i' \in \mathbb{Z}_n \times \{0, 1, 2\}$.
$S_5 = \{T_1^5, T_2^5, \dots, T_{m-1}^5\}$.

(6) $|S_6| = 3$.
$S_6 = \{\{\infty, (f, 2), (f+m, 2)\}, \{\beta, (g, 0), (g+m, 0)\}, \{\alpha, (h, 1), (h+m, 1)\}$.
$S_6 = \{T_1^6, T_2^6, T_3^6\}$.

The starter $S = \{S_1, S_2, S_3, S_4, S_5, S_6\}$ has the following properties.

(1) $S_1 \cup S_2 \cup S_3 = V$.

(2) Let $S_5 + m = \{T_1^5 + m, T_2^5 + m, \dots, T_{m-1}^5 + m\}$.
$S_5 \cup (S_5 + m) \cup S_6 = V$.

(3) Every nonzero element of $\mathbb{Z}_n$ occurs precisely once in S as a pure (j, j) difference for $j = 0, 1, 2$.

(4) Every element of $\mathbb{Z}_n$ occurs precisely once in S as a mixed (i, j) difference for $i \neq j$ and $i, j = 0, 1, 2$.

A corresponding adder $A(S)$ consists of four sets, $A(S_1)$, $A(S_3)$, $A(S_5)$, $A(S_6)$ of the following form.

(1) $A(S_1) = (a_1, a_2, \dots, a_{m+1})$ where $a_i \in \mathbb{Z}_n$ and no element of $\mathbb{Z}_n$ occurs more than once in $A(S_1)$.

(2) $A(S_3) = (b_1, b_2, \dots, b_{m-1})$ where $b_i \neq b_j$ for all $i \neq j$ and $b_i \in \{0, 1, \dots, m-1\}$.

(3) $A(S_5) = (c_1, c_2, \ldots, c_{m-1})$ where $c_i \neq c_j$ for all $i \neq j$, $c_i \in \mathbb{Z}_n$, and $|c_i - c_j| \neq m$ for all i, j.

(4) $A(S_6) = (d_1, d_2, d_3)$ where $d_i \neq d_j$ for all $i \neq j$ and $d_i \in \{0, 1, \ldots, m-1\}$.

$A(S)$ has the following properties.

(1) $(S_1 + A(S_1)) \cup S_4 \cup (S_5 + A(S_5)) = V$.

(2) $(S_3 + A(S_3)) \cup (S_3 + (A(S_3) + m)) \cup (S_6 + A(S_6)) = V$.

THEOREM 3.4. *If there exists a starter S for a* $KS_3(3n+3; 1, 1)$ *defined on* $(\mathbb{Z}_n \times \{0, 1, 2\}) \cup \{\infty, \beta, \alpha\}$ *and a corresponding adder* $A(S)$, *then there exists a* $KS_3(3n+3; 1, 1)$ *with an automorphism group which partitions the rows and columns into orbits of lengths n,* $n/2$, *and* 1.

Proof. Let $n = 2m$. Let $V = (\mathbb{Z}_n \times \{0, 1, 2\}) \cup \{\infty, \alpha, \beta\}$. We use S and the corresponding adder $A(S)$ to construct 6 arrays. (All arithmetic is done modulo n.)

We first construct an $n \times n$ array K_1 from S_1 and $A(S_1)$. Label the columns of K_1 by $0, n-1, n-2, \ldots, 2, 1$ and the rows by $0, 1, 2, \ldots, n-1$. In row 0, column a_i, place the triple T_i^1. The rest of the array is generated cyclically from row 0. In row j, column $a_i - j$, place $T_i^1 + j$.

S_2 generates an $n \times 1$ array, K_2; row j contains $S_2 + j$. Similarly, S_4 generates a $1 \times n$ array, K_4; column j contains $S_4 + j$.

Next we use S_3 and $A(S_3)$ to generate an $n \times m$ array K_3. Label the rows of K_3 $0, 1, \ldots, n-1$ and the columns $0, m-1, m-2, \ldots, 2, 1$. In row j, column $b_i - j$, place $T_i^3 + j$ for $j = 0, 1, \ldots, m-1$. In row j, column $b_i + m - j$, place $T_i^3 + j$ for $j = m, m+1, \ldots, n-1$. (The array is still generated cyclically.)

We construct an $m \times n$ array K_5 from S_5 and $A(S_5)$. Label the columns of K_5 $0, n-1, n-2, \ldots, 2, 1$ and the rows $0, 1, 2, \ldots, m-1$. In row 0, column c_i, place T_i^5, and in row 0, column $c_i + m$, place $T_i^5 + m$, for $i = 0, 1, \ldots, m-1$. (So row 0 now contains $n-2$ triples.) In row j, column $c_i - j$, place $T_i^5 + j$ and in column $c_i + m - j$, place $T_i^5 + m + j$ for $j = 1, 2, \ldots, m-1$.

Finally, we construct an $m \times m$ array K_6 from the three triples in S_6. Again, we label the rows $0, 1, \ldots, m-1$ and the columns $0, m-1, m-2, \ldots, 1$. In row j, column $d_i - j$, place $T_i^6 + j$.

We construct an array K from $K_1, \ldots, K_6$ as follows. Let $X = \{\infty, \alpha, \beta\}$.

K_1	K_2	K_3
K_4	X	
K_5		K_6

K

We claim that the array K is a $KS_3(3n+3; 1, 1)$. The structure and properties of the starter S ensure that every element of V occurs precisely once in each row of K. The properties of the corresponding adder $A(S)$ ensure that every element of V occurs once in each column. It is straightforward to verify that the underlying design is a $(3n+3, 3, 1)$-*BIBD*. Finally, the construction of K displays the partitioning of the rows and columns into orbits of lengths n, $m = n/2$, and 1. ∎

LEMMA 3.5. *There exists a* $KS_3(33; 1, 1)$.

Proof. Let $V = (\mathbb{Z}_{10} \times \{0, 1, 2\}) \cup \{\infty, \beta, \alpha\}$. A starter and corresponding adder are shown in Table 3. ∎

This $KS_3(33; 1, 1)$ is displayed in Figure 2.

LEMMA 3.6. *There exists a* $KS_3(45; 1, 1)$.

Proof. Let $V = (\mathbb{Z}_{14} \times \{0, 1, 2\}) \cup \{\infty, \beta, \alpha\}$. A starter and corresponding adder are shown in Table 4. ∎

The last construction in this section is a tripling construction which uses cyclically generated $(p, 3, 1)$-*BIBD*s ([8]) and a set of three mutually orthogonal Latin squares of order p to construct a $KS_3(3p; 1, 1)$ where $p = 6t + 1$.

LEMMA 3.7. *There exists a* $KS_3(3p; 1, 1)$ *for* $p = 37, 43$.

Table 3. Starter and adder for a $KS_3(33; 1, 1)$.

S_1	$\{(2,0), (6,0), (1,1)\}$	0
	$\{\infty, (8,0), (5,1)\}$	7
	$\{\beta, (9,1), (0,2)\}$	5
	$\{\alpha, (4,0), (9,2)\}$	4
	$\{(9,0), (5,2), (7,2)\}$	2
	$\{(2,1), (8,1), (1,2)\}$	1
S_2	$\{(7,0), (0,1), (4,2)\}$	
S_3	$\{(0,0), (1,0), (3,0)\}$	0
	$\{(5,0), (3,1), (8,2)\}$	4
	$\{(2,2), (3,2), (6,2)\}$	3
	$\{(4,1), (6,1), (7,1)\}$	2
S_4	$\{(0,0), (0,1), (0,2)\}$	
S_5	$\{(0,0), (1,1), (9,2)\}$	7
	$\{(1,0), (7,1), (3,2)\}$	8
	$\{(3,0), (5,1), (7,2)\}$	1
	$\{(4,0), (8,1), (5,2)\}$	9
S_6	$\{\beta, (2,0), (7,0)\}$	0
	$\{\infty, (1,2), (6,2)\}$	2
	$\{\alpha, (4,1), (9,1)\}$	1

$2_0 6_0 1_1$			$Y8_0 5_1$		$X9_1 0_2$	$Z4_0 9_2$		$9_0 5_2 7_2$	$2_1 8_1 1_2$	$7_0 0_1 4_2$	$0_0 1_0 3_0$	$5_0 3_1 8_2$	$2_2 3_2 6_2$	$4_1 6_1 7_1$	
$3_1 9_1 2_2$	$3_0 7_0 2_1$			$Y9_0 6_1$		$X0_1 1_2$	$Z5_0 0_2$		$0_0 6_2 8_2$	$8_0 1_1 5_2$		$1_0 2_0 4_0$	$6_0 4_1 9_2$	$3_2 4_2 7_2$	$5_1 7_1 8_1$
$1_0 7_2 9_2$	$4_1 0_1 3_2$	$4_0 8_0 3_1$			$Y0_0 7_1$		$X1_1 2_2$	$Z6_0 1_2$		$9_0 2_1 6_2$	$6_1 8_1 9_1$		$2_0 3_0 5_0$	$7_0 5_1 0_2$	$4_2 5_2 8_2$
	$2_0 8_2 0_2$	$5_1 1_1 4_2$	$5_0 9_0 4_1$			$Y1_0 8_1$		$X2_1 3_2$	$Z7_0 2_2$	$0_0 3_1 7_2$	$5_2 6_2 9_2$	$7_1 9_1 0_1$		$3_0 4_0 6_0$	$8_0 6_1 1_2$
$Z8_0 3_2$		$3_0 9_2 1_2$	$6_1 2_1 5_2$	$6_0 0_0 5_1$			$Y2_0 9_1$		$X3_1 4_2$	$1_0 4_1 8_2$	$9_0 7_1 2_2$	$6_2 7_2 0_2$	$8_1 0_1 1_1$		$4_0 5_0 7_0$
$X4_1 5_2$	$Z9_0 4_2$		$4_0 0_2 2_2$	$7_1 3_1 6_2$	$7_0 1_0 6_1$			$Y3_0 0_1$		$2_0 5_1 9_2$	$5_0 6_0 8_0$	$0_0 8_1 3_2$	$7_2 8_2 1_2$	$9_1 1_1 2_1$	
	$X5_1 6_2$	$Z0_0 5_2$		$5_0 1_2 3_2$	$8_1 4_1 7_2$	$8_0 2_0 7_1$			$Y4_0 1_1$	$3_0 6_1 0_2$		$6_0 7_0 9_0$	$1_0 9_1 4_2$	$8_2 9_2 2_2$	$0_1 2_1 3_1$
$Y5_0 2_1$		$X6_1 7_2$	$Z1_0 6_2$		$6_0 2_2 4_2$	$9_1 5_1 8_2$	$9_0 3_0 8_1$			$4_0 7_1 1_2$	$1_1 3_1 4_1$		$7_0 8_0 0_0$	$2_0 0_1 5_2$	$9_2 0_2 3_2$
	$Y6_0 3_1$		$X7_1 8_2$	$Z2_0 7_2$		$7_0 3_2 5_2$	$0_1 6_1 9_2$	$0_0 4_0 9_1$		$5_0 8_1 2_2$	$0_2 1_2 4_2$	$2_1 4_1 5_1$		$8_0 9_0 1_0$	$3_0 1_1 6_2$
		$Y7_0 4_1$		$X8_1 9_2$	$Z3_0 8_2$		$8_0 4_2 6_2$	$1_1 7_1 0_2$	$1_0 5_0 0_1$	$6_0 9_1 3_2$	$4_0 2_1 7_2$	$1_2 2_2 5_2$	$3_1 5_1 6_1$		$9_0 0_0 2_0$
$0_0 0_1 0_2$	$1_0 1_1 1_2$	$2_0 2_1 2_2$	$3_0 3_1 3_2$	$4_0 4_1 4_2$	$5_0 5_1 5_2$	$6_0 6_1 6_2$	$7_0 7_1 7_2$	$8_0 8_1 8_2$	$9_0 9_1 9_2$	XYZ					
	$4_0 8_1 5_2$	$1_0 7_1 3_2$	$0_0 1_1 9_2$	$8_0 0_1 2_2$		$9_0 3_1 0_2$	$6_0 2_1 8_2$	$5_0 6_1 4_2$	$3_0 5_1 7_2$		$X2_0 7_0$			$Y1_2 6_2$	$Z4_1 9_1$
$4_0 6_1 8_2$		$5_0 9_1 6_2$	$2_0 8_1 4_2$	$1_0 2_1 0_2$	$9_0 1_1 3_2$		$0_0 4_1 1_2$	$7_0 3_1 9_2$	$6_0 7_1 5_2$		$Z5_1 0_1$	$X3_0 8_0$			$Y2_2 7_2$
$7_0 8_1 6_2$	$5_0 7_1 9_2$		$6_0 0_1 7_2$	$3_0 9_1 5_2$	$2_0 3_1 1_2$	$0_0 2_1 4_2$		$1_0 5_1 2_2$	$8_0 4_1 0_2$		$Y3_2 8_2$	$Z6_1 1_1$	$X4_0 9_0$		
$9_0 5_1 1_2$	$8_0 9_1 7_2$	$6_0 8_1 0_2$		$7_0 1_1 8_2$	$4_0 0_1 6_2$	$3_0 4_1 2_2$	$1_0 3_1 5_2$		$2_0 6_1 3_2$			$Y4_2 9_2$	$Z7_1 2_1$	$X5_0 0_0$	
$3_0 7_1 4_2$	$0_0 6_1 2_2$	$9_0 0_1 8_2$	$7_0 9_1 1_2$		$8_0 2_1 9_2$	$5_0 1_1 7_2$	$4_0 5_1 3_2$	$2_0 4_1 6_2$					$Y5_2 0_2$	$Z8_1 3_1$	$X6_0 1_0$

Figure 2. A $KS_3(33; 1, 1)$.

Proof. Let $p = 6t + 1$. The number of resolution classes in a resolvable $(3p, 3, 1)$-*BIBD* is $p + 3t$. Let $V = \mathbb{Z}_p \times \{0, 1, 2\}$. Let ω be a generator of $\mathbb{Z}_p$. We construct a square of $p + 3t$ by $p + 3t$ of the form

A	X
Y	E

where A is a $p \times p$ square, X is $p \times 3t$, Y is $3t \times p$, and E is a $3t \times 3t$ empty square. X is determined as follows. In the first row place the block

$$\{(\omega^{2i}, 0), (\omega^{2t+2i}, 1), (\omega^{4t+2i}, 2)\}$$

Table 4. Starter and adder for a $KS_3(45; 1, 1)$.

S_1	$\{(5,0),(10,0),(7,1)\}$	3
	$\{(13,0),(7,2),(11,2)\}$	10
	$\{(12,0),(11,1),(5,2)\}$	4
	$\{(11,0),(6,1),(10,2)\}$	6
	$\{(9,1),(13,1),(6,2)\}$	12
	$\{\infty,(9,0),(5,1)\}$	1
	$\{\alpha,(7,0),(9,2)\}$	11
	$\{\beta,(10,1),(12,2)\}$	7
S_2	$\{(8,0),(12,1),(3,2)\}$	
S_3	$\{(0,0),(1,0),(3,0)\}$	0
	$\{(2,0),(6,0),(0,1)\}$	3
	$\{(4,0),(0,2),(1,2)\}$	2
	$\{(1,1),(2,1),(4,1)\}$	4
	$\{(3,1),(8,1),(4,2)\}$	1
	$\{(2,2),(8,2),(13,2)\}$	5
S_4	$\{(0,0),(0,1),(0,2)\}$	
S_5	$\{(0,0),(6,0),(7,1)\}$	1
	$\{(5,1),(13,1),(11,2)\}$	0
	$\{(3,0),(6,2),(8,2)\}$	2
	$\{(1,0),(4,1),(7,2)\}$	5
	$\{(2,0),(8,1),(3,2)\}$	10
	$\{(5,0),(10,1),(9,2)\}$	6
S_6	$\{\beta,(4,0),(11,0)\}$	0
	$\{\alpha,(2,1),(9,1)\}$	5
	$\{\infty,(5,2),(12,2)\}$	3

for $i = 0, 1, 2, \ldots, 3t-1$. Hence the first row of X is full. The rest of X is developed cyclically modulo p through the rows: for $j = 0, 1, \ldots, p-1$, row $j+1$ contains the blocks $\{(\omega^{2i}+j, 0), (\omega^{2t+2i}+j, 1), (\omega^{4t+2i}+j, 2)\}$ for $i = 0, 1, 2, \ldots, 3t-1$ where all arithmetic is done modulo p.

Y is determined as follows. In the first column place the block

$$\{(\omega^{2i+1}, 0), (\omega^{2t+2i+1}, 1), (\omega^{4t+2i+1}, 2)\}$$

for $i = 0, 1, 2, \ldots, 3t-1$. Hence the first column of Y is full. The rest of Y is developed cyclically modulo p through the columns: for $j = 0, 1, \ldots, p-1$, column $j+1$ contains the blocks $\{(\omega^{2i+1}+j, 0), (\omega^{2t+2i+1}+j, 1), (\omega^{4t+2i+1}+j, 2)\}$ for $i = 0, 1, 2, \ldots, 3t-1$ where all arithmetic is done modulo p.

For $p = 37$, the first row of A is constructed as shown in Table 5; when $\{x, y, z\}, c$ is written, the triple $\{x, y, z\}$ is placed in the cell $(0, c)$ with columns indexed from 0 to $p-1$. The remainder of A is developed cyclically through rows and columns. For $p = 43$, the first row of A is constructed as in Table 6. ∎

The following theorem summarizes the results of this section.

THEOREM 3.8. *There exist* $KS_3(6x+3; 1, 1)$ *for* $x = 5, 7, 12, 14, 18, 20, 21, 22, 26, 34, 36, 42, 46, 50, 52, 66, 74$.

Table 5. Construction for 3 · 37.

{(0, 0), (0, 1), (0, 2)}, 0	
{(2, 0), (5, 0), (17, 0)}, 1	{(6, 0), (14, 0), (32, 0)}, 7
{(13, 0), (20, 0), (22, 0)}, 10	{(15, 0), (19, 0), (35, 0)}, 26
{(23, 0), (24, 0), (29, 0)}, 33	{(8, 0), (18, 0), (31, 0)}, 34
{(5, 1), (29, 1), (35, 1)}, 2	{(2, 1), (23, 1), (24, 1)}, 14
{(14, 1), (19, 1), (22, 1)}, 15	{(13, 1), (17, 1), (31, 1)}, 20
{(8, 1), (18, 1), (20, 1)}, 29	{(6, 1), (15, 1), (32, 1)}, 31
{(6, 2), (14, 2), (15, 2)}, 3	{(8, 2), (20, 2), (31, 2)}, 4
{(5, 2), (18, 2), (22, 2)}, 21	{(13, 2), (32, 2), (35, 2)}, 25
{(17, 2), (19, 2), (24, 2)}, 28	{(2, 2), (23, 2), (29, 2)}, 30

Table 6. Construction for 3 · 43.

{(0, 0), (0, 1), (0, 2)}, 0	{(29, 2), (37, 2), (42, 2)}, 36
{(2, 0), (7, 0), (37, 0)}, 1	{(8, 0), (19, 0), (28, 0)}, 4
{(20, 0), (22, 0), (34, 0)}, 11	{(26, 0), (32, 0), (33, 0)}, 16
{(3, 0), (18, 0), (42, 0)}, 21	{(5, 0), (27, 0), (30, 0)}, 35
{(12, 0), (29, 0), (39, 0)}, 41	{(7, 1), (12, 1), (42, 1)}, 6
{(20, 1), (26, 1), (27, 1)}, 10	{(3, 1), (32, 1), (34, 1)}, 23
{(5, 1), (28, 1), (39, 1)}, 24	{(2, 1), (19, 1), (29, 1)}, 31
{(8, 1), (30, 1), (33, 1)}, 38	{(18, 1), (22, 1), (37, 1)}, 40
{(18, 2), (20, 2), (32, 2)}, 9	{(5, 2), (8, 2), (26, 2)}, 13
{(2, 2), (12, 2), (28, 2)}, 14	{(19, 2), (30, 2), (39, 2)}, 15
{(27, 2), (33, 2), (34, 2)}, 17	{(3, 2), (7, 2), (22, 2)}, 25

4. Frames

Our main recursive construction uses frames. In order to describe this construction, we need several definitions. A group divisible design (GDD) of index λ is a triple $(\mathcal{X}, \mathcal{G}, \mathcal{B})$ which satisfies the following properties.

(1) $\mathcal{G}$ is a partition of $\mathcal{X}$ into subsets called groups; $\mathcal{G} = \{G_1, G_2, \ldots, G_m\}$.

(2) $\mathcal{B}$ is a collection of subsets of $\mathcal{X}$, called blocks, such that a group and a block contain at most one element in common.

(3) Every pair of elements from distinct groups occurs in precisely λ blocks.

A *GDD*$(v; K; G_1, G_2, \ldots, G_m; 0, \lambda)$ G is a group divisible design of index λ with $|\mathcal{X}| = v$, $|b| \in K$ for every $b \in \mathcal{B}$, and $\mathcal{G} = \{G_1, G_2, \ldots, G_m\}$. The type of G is the multiset $\{|G_1|, |G_2|, \ldots, |G_m|\}$. We usually use exponential notation to describe the type; G has type $t_1^{u_1} t_2^{u_2} \ldots t_\ell^{u_\ell}$ if there are u_i G_j's of cardinality t_i, $1 \le i \le \ell$. A *GDD*$(v; K; G_1, \ldots, G_m; 0, 1)$ is often denoted as a K-*GDD* of type $t_1^{u_1} t_2^{u_2} \ldots t_\ell^{u_\ell}$. A transversal design, *TD*(k, n), is a *GDD* which has k groups of size n and block size k ($\lambda = 1$) or a $\{k\}$-*GDD* of type n^k. It is well known that a *TD*$(k + 2, n)$ is equivalent to the existence of a set of k mutually orthogonal Latin squares of side n. For existence results on *TD*(k, n), we refer to [2].

Let V be a set of v elements. Let $G_1, G_2, \ldots, G_m$ be a partition of V into m sets. A $\{G_1, G_2, \ldots, G_m\}$-frame F with block size k, index λ, and latinicity μ is a square array of side $t = (\lambda v)/(\mu(k-1))$ which satisfies the properties listed below. Let $t_i = (\lambda|G_i|)/(\mu(k-1))$, and let $g_k = \sum_{i=1}^{k} t_i$ and define $g_0 = 0$. We index the rows and columns of F with the elements $0, 1, \ldots, (\lambda v)/(\mu(k-1)) - 1$.

(1) Each cell is either empty or contains a k-subset of V.

(2) Let F_i be the subsquare of F indexed by $g_{i-1}, g_{i-1}+1, \ldots, g_i - 1$. F_i is empty for $i = 1, 2, \ldots, m$. (i.e., The main diagonal of F consists of empty subsquares of sides $t_i \times t_i$ for $i = 1, 2, \ldots, m$.)

(3) Let $x \in \{g_{i-1}, g_{i-1}+1, \ldots, g_i - 1\}$. Row x of F contains each element of $V - G_i\,\mu$ times and no element of G_i. Similarly, column x of F contains each element of $V - G_i\,\mu$ times and no element of G_i.

(4) The blocks obtained from the nonempty cells of F form a *GDD*$(v; k; G_1, G_2, \ldots, G_m; 0, \lambda)$.

The type of a $\{G_1, G_2, \ldots, G_m\}$-frame is the multi-set $\{|G_1|, |G_2|, \ldots, |G_m|\}$. We usually use exponential notation to describe the type; a frame has type $t_1^{u_1} t_2^{u_2} \ldots t_\ell^{u_\ell}$ if there are u_i G_j's of cardinality t_i, $1 \le i \le \ell$. A $\{G_1, G_2, \ldots, G_m\}$-frame with block size k, index λ, and latinicity μ is denoted as a $(\mu, \lambda; k, \{G_1, \ldots, \mathcal{G}_m\})$-frame or a $(\mu, \lambda; k)$-frame of type $t_1^{u_1} t_2^{u_2} \ldots t_\ell^{u_\ell}$ if there are u_i G_j's of cardinality t_i, $1 \le i \le \ell$.

The majority of our constructions in the next section use some variation of the Basic Frame Construction. (The proof is similar to that of [26, Theorem 3.6]).

THEOREM 4.1 (Basic Frame Construction). *Suppose there exists a* $(1, 1; 3, \{G_1, G_2, \ldots, G_m\})$*-frame. If there exists a* $KS_3(|G_i|+u; 1, 1)$ *which contains as a subarray a* $KS_3(u; 1, 1)$ *for all i,* $1 \le i \le m$*, then there is a* $KS_3(\sum_{i=1}^{m} |G_i| + u; 1, 1)$ *which contains as a subarray a* $KS_3(u; 1, 1)$.

In order to apply the Basic Frame Construction, we need some existence results for $(1, 1; 3, \{G_1, G_2, \ldots, G_m\})$-frames. Two standard recursive constructions can be used to construct $(1, 1; 3)$-frames. The proofs are similar to those in [10,21,46].

THEOREM 4.2 (Singular Direct Product). *If there exists a* $(1, 1; 3, \{G_1, G_2, \ldots, G_m\})$*-frame and a set of 3 mutually orthogonal Latin squares of side n, then there exists a* $(1, 1; 3)$*-frame of type* $\{n|G_i| \mid i = 1, 2, \ldots, m\}$.

THEOREM 4.3 (Fundamental Construction). *Let G be a* $GDD(v; K; G_1, G_2, \ldots, G_m; 0, 1)$. *Suppose there exists a function* $w : V \to \mathbb{Z}^+ \cup \{0\}$ *(a weight function) which has the property that for each block* $b = \{x_1, x_2, \ldots, x_k\} \in \mathcal{B}$*, there exists a* $(1, 1; 3)$*-frame of type* $\{w(x_1), w(x_2), \ldots, w(x_k)\}$*. Then there exists a* $(1, 1; 3)$*-frame of type*

$$\left\{ \sum_{x \in G_1} w(x), \sum_{x \in G_2} w(x), \ldots, \sum_{x \in G_m} w(x) \right\}.$$

Let V be a finite set of cardinality v. A $KS_3(v+1; 1, 1)$ defined on $V \cup \{\infty\}$ is said to be in standard form if the element ∞ occurs in each cell on the main diagonal. If we delete the triples on the main diagonal of a $KS_3(v+1; 1, 1)$ in standard form, the resulting design is a $(1, 1; 3)$-frame of type $2^{(v/2)}$. Since every $KS_3(v+1; 1, 1)$ can be put in standard form, we have the following.

LEMMA 4.4. *If there exists a $KS_3(v+1; 1, 1)$, then there is a $(1, 1; 3)$-frame of type $2^{v/2}$.*

The next result is used for several small designs in the next section.

LEMMA 4.5 [10]. *There exist $(1, 1; 3)$-frames of types 4^7 and 4^{10}.*

The $(1, 1; 3, \{G_1, G_2, \ldots, G_m\})$-frames which are most useful to us in the recursive constructions are of type 6^n. We use starters and adders to construct four frames.

Let $m = 2n + 1$, and let $\mathbb{Z}^*_{3m} = \mathbb{Z}_{3m} - \{0, m, 2m\}$. A frame starter S over $\mathbb{Z}_{3m} \times \{0, 1\}$ for a $(1, 1; 3)$-frame of type 6^m with groups

$$\{(i, 0), (i+m, 0), (i+2m, 0), (i, 1), (i+m, 1), (i+2m, 1)\}$$

for $i = 0, 1, \ldots, m-1$ is a collection of $4n$ triples $S = \{S_1, S_2, \ldots, S_{4n}\}$ which satisfy the following properties.

(1) $\cup_{i=1}^{4n} S_i = \mathbb{Z}^*_{3m} \times \{0, 1\}$.

(2) Every element of $\mathbb{Z}^*_{3m}$ occurs precisely once in S as a pure (j, j) difference for $j = 0, 1$.

(3) Every element of $\mathbb{Z}^*_{3m}$ occurs precisely once in S as a mixed (i, j) difference for $i \neq j$ and $i, j = 0, 1$.

Let $A(S) = (a_1, a_2, \ldots, a_{4n})$ be a set of $4n$ elements of $\mathbb{Z}^*_{3m}$ such that no element of $\mathbb{Z}^*_{3m}$ occurs more than once in $A(S)$. $A(S)$ is an adder for the frame starter S if $\cup_{i=1}^{4n}(S_i + a_i) = \mathbb{Z}^*_{3m} \times \{0, 1\}$.

THEOREM 4.6. *Let $m = 2n + 1$. If there exists a frame starter S over $\mathbb{Z}_{3m} \times \{0, 1\}$ for a $(1, 1; 3)$-frame of type 6^m and a corresponding adder $A(S)$, then there is a $(1, 1; 3)$-frame of type 6^m.*

Proof. This proof is similar to the proof of Theorem 3.1. We construct a $3m \times 3m$ array using a frame starter S and a corresponding adder $A(S)$. As before, we label the rows $0, 1, \ldots, 3m-1$ and the columns $0, 3m-1, \ldots, 2, 1$. In row j and column $a_i - j$, place the triple $S_i + j$ for all $i, j = 1, 2, \ldots, 3m-1$. It is straightforward to verify that this produces a $(1, 1; 3)$-frame of type 6^m. ∎

LEMMA 4.7. *There exists a $(1, 1; 3)$-frame of type 6^7.*

Proof. Let $V = \mathbb{Z}_{21} \times \{0, 1\}$. The groups are

$$\{(i, 0), (i+7, 0), (i+14, 0), (i, 1), (i+7, 1), (i+14, 1)\}$$

for $i = 0, 1, \ldots, 6$. The starter blocks are as shown in Table 7. ∎

Table 7. Frame of type 6^7.

Base Block (First Resolution)	Adder	Base Block + Adder (Orthogonal Resolution)
{(1, 0), (2, 0), (10, 0)}	1	{(2, 0), (3, 0), (11, 0)}
{(4, 0), (6, 0), (9, 0)}	9	{(13, 0), (15, 0), (18, 0)}
{(5, 0), (11, 0), (15, 0)}	11	{(16, 0), (1, 0), (5, 0)}
{(3, 0), (5, 1), (13, 1)}	3	{(6, 0), (8, 1), (16, 1)}
{(8, 0), (3, 1), (4, 1)}	2	{(10, 0), (5, 1), (6, 1)}
{(12, 0), (9, 1), (18, 1)}	13	{(4, 0), (1, 1), (10, 1)}
{(13, 0), (1, 1), (16, 1)}	17	{(9, 0), (18, 1), (12, 1)}
{(16, 0), (15, 1), (20, 1)}	4	{(20, 0), (19, 1), (3, 1)}
{(17, 0), (8, 1), (11, 1)}	12	{(8, 0), (20, 1), (2, 1)}
{(18, 0), (2, 1), (19, 1)}	15	{(12, 0), (17, 1), (13, 1)}
{(19, 0), (6, 1), (17, 1)}	19	{(17, 0), (4, 1), (15, 1)}
{(20, 0), (10, 1), (12, 1)}	20	{(19, 0), (9, 1), (11, 1)}

LEMMA 4.8. *There exists a* $(1, 1; 3)$*-frame of type* 6^9.

Proof. Let $V = \mathbb{Z}_{27} \times \{0, 1\}$. The groups are

$$\{(i, 0), (i + 9, 0), (i + 18, 0), (i, 1), (i + 9, 1), (i + 18, 1)\}$$

for $i = 0, 1, \ldots, 8$. The starter blocks are as shown in Table 8. ■

LEMMA 4.9. *There exists a* $(1, 1; 3)$*-frame of type* 6^{13}.

Proof. Let $V = \mathbb{Z}_{39} \times \{0, 1\}$. The groups are

$$\{(i, 0), (i + 13, 0), (i + 26, 0), (i, 1), (i + 13, 1), (i + 26, 1)\}$$

for $i = 0, 1, \ldots, 12$. Eight starter blocks and their corresponding adders are shown in Table 9. The remaining 16 starter blocks and their corresponding adders are obtained by multiplying each of the eight starter blocks (and their adders) in Table 9 by 16 and 22 modulo 39. ■

LEMMA 4.10. *There exists a* $(1, 1; 3)$*-frame of type* 6^{19}.

Proof. Let $V = \mathbb{Z}_{57} \times \{0, 1\}$. The groups are

$$\{(i, 0), (i + 19, 0), (i + 38, 0), (i, 1), (i + 19, 1), (i + 38, 1)\}$$

for $i = 0, 1, \ldots, 18$. Twelve starter blocks and their corresponding adders are shown in Table 10. The remaining 24 starter blocks and their corresponding adders are obtained by

Table 8. Frame of type 6^9.

Base Block (First Resolution)	Adder	Base Block + Adder (Orthogonal Resolution)
{(14, 1), (1, 0), (2, 0)}	3	{(17, 1), (4, 0), (5, 0)}
{(14, 0), (1, 1), (2, 1)}	6	{(20, 0), (7, 1), (8, 1)}
{(15, 1), (17, 1), (23, 1)}	5	{(20, 1), (22, 1), (1, 1)}
{(15, 0), (17, 0), (23, 0)}	20	{(8, 0), (10, 0), (16, 0)}
{(16, 1), (13, 0), (20, 0)}	13	{(2, 1), (26, 0), (6, 0)}
{(16, 0), (13, 1), (20, 1)}	12	{(1, 0), (25, 1), (5, 1)}
{(3, 0), (5, 1), (8, 1)}	8	{(11, 0), (13, 1), (16, 1)}
{(3, 1), (5, 0), (8, 0)}	16	{(19, 1), (21, 0), (24, 0)}
{(4, 0), (25, 1), (12, 1)}	26	{(3, 0), (24, 1), (11, 1)}
{(4, 1), (25, 0), (12, 0)}	17	{(21, 1), (15, 0), (2, 0)}
{(19, 1), (24, 1), (7, 1)}	7	{(26, 1), (4, 1), (14, 1)}
{(19, 0), (24, 0), (7, 0)}	15	{(7, 0), (12, 0), (22, 0)}
{(6, 0), (22, 1), (26, 1)}	11	{(17, 0), (6, 1), (10, 1)}
{(6, 1), (22, 0), (26, 0)}	24	{(3, 1), (19, 0), (23, 0)}
{(21, 1), (11, 0), (10, 1)}	2	{(23, 1), (13, 0), (12, 1)}
{(21, 0), (11, 1), (10, 0)}	4	{(25, 0), (15, 1), (14, 0)}

Table 9. Frame of type 6^{13}.

Base Block (First Resolution)	Adder	Base Block + Adder (Orthogonal Resolution)
{(1, 0), (2, 1), (3, 1)}	1	{(2, 0), (3, 1), (4, 1)}
{(2, 0), (1, 1), (6, 1)}	5	{(7, 0), (6, 1), (11, 1)}
{(3, 0), (10, 1), (31, 1)}	6	{(9, 0), (16, 1), (37, 1)}
{(4, 0), (36, 0), (12, 1)}	11	{(15, 0), (8, 0), (23, 1)}
{(8, 0), (28, 0), (37, 1)}	21	{(29, 0), (10, 0), (19, 1)}
{(14, 0), (17, 0), (11, 1)}	19	{(33, 0), (36, 0), (30, 1)}
{(6, 0), (7, 0), (21, 0)}	10	{(16, 0), (17, 0), (31, 0)}
{(14, 1), (23, 1), (33, 1)}	30	{(5, 1), (14, 1), 24, 1)}

multiplying each of the twelve starter blocks (and their adders) in Table 10 by 7 and 49 modulo 57. ■

Intransitive starters and adders are used for three other frames. An intransitive starter over $\mathbb{Z}_{3m} \times \{0, 1\}$ for a (1, 1; 3)-frame of type $6^m(6k)^1$ written on the symbol set $(\mathbb{Z}_{3m} \cup \{\infty_0, \infty_1, \ldots, \infty_{3k-1}\}) \times \{0, 1\}$ with groups

$$G_i = \{(i, 0), (i + m, 0), (i + 2m, 0), (i, 1), (i + m, 1), (i + 2m, 1)\}$$

Table 10. Frame of type 6^{19}.

Base Block (First Resolution)	Adder	Base Block + Adder (Orthogonal Resolution)
{(1, 0), (2, 1), (3, 1)}	1	{(2, 0), (3, 1), (4, 1)}
{(2, 0), (1, 1), (5, 1)}	4	{(6, 0), (5, 1), (9, 1)}
{(3, 0), (8, 1), (13, 1)}	2	{(5, 0), (10, 1), (15, 1)}
{(4, 0), (5, 0), (16, 1)}	6	{(10, 0), (11, 0), (22, 1)}
{(6, 0), (8, 0), (4, 1)}	23	{(29, 0), (31, 0), (27, 1)}
{(10, 0), (36, 0), (40, 1)}	46	{(56, 0), (25, 0), (29, 1)}
{(11, 0), (39, 0), (26, 1)}	5	{(16, 0), (44, 0), (31, 1)}
{(12, 0), (51, 0), (18, 1)}	36	{(48, 0), (30, 0), (54, 1)}
{(16, 0), (52, 0), (47, 1)}	8	{(24, 0), (3, 0), (55, 1)}
{(23, 0), (29, 0), (46, 0)}	29	{(52, 0), (1, 0), (18, 0)}
{(6, 1), (37, 1), (53, 1)}	43	{(49, 1), (23, 1), (39, 1)}
{(15, 1), (30, 1), (54, 1)}	11	{(26, 1), (41, 1), (8, 1)}

for $i = 0, 1, \ldots, m-1$, and

$$G_m = \{(\infty_i, 0), (\infty_i, 1) \mid i = 0, 1, \ldots, 3k-1\}$$

is defined to be a triple (S, R, C) where

(1) $S = \{\{y_i, z_i, w_i\} \mid i = 1, 2, \ldots, 2m-2-6k\} \cup$
$\{\{(\infty_i, 0), u_i, v_i\}, \{(\infty_i, 1), s_i, t_i\} \mid i = 0, 1, \ldots, 3k-1\}$.

(2) $C = \{\{c_{i1}, c_{i2}, c_{i3}\}, \{c'_{i1}, c'_{i2}, c'_{i3}\} \mid i = 1, 2, \ldots, k\}$.

(3) $R = \{\{d_{i1}, d_{i2}, d_{i3}\}, \{d'_{i1}, d'_{i2}, d'_{i3}\} \mid i = 1, 2, \ldots, k\}$.

where $y_i, z_i, w_i, u_i, v_i, s_i, t_i \in \mathbb{Z}^*_{3m} \times \{0, 1\}$ and $c_{ij}, d_{ij}, c'_{ij}, d'_{ij} \in \mathbb{Z}^*_{3m} \times \{0, 1\}$ for $j = 1, 2, 3$ and $i = 1, 2, \ldots, k$. (S, R, C) satisfies the following properties.

(1) $S \cup C = (\mathbb{Z}^*_{3m} \cup \{(\infty_i, 0), (\infty_i, 1) \mid i = 0, 1, \ldots, 3k-1\}) \times \{0, 1\}$.

(2) Every element of $\mathbb{Z}^*_{3m}$ occurs precisely once as a pure (j, j) difference in (S, R, C) for $j = 0, 1$.

(3) Every element of $\mathbb{Z}^*_{3m}$ occurs precisely once as a mixed (i, j) difference in a triple in (S, R, C) for $i \neq j$ and $i, j = 0, 1$.

(4) The triples in C contain no difference (pure or mixed) congruent to 0 modulo 3.

(5) The triples in R contain no difference (pure or mixed) congruent to 0 modulo 3.

(6) The triples in C can be paired, $\{c_{i1}, c_{i1}, c_{i3}\}$ and $\{c'_{i1}, c'_{i2}, c'_{i3}\}$, so that each pair contains three elements from $\mathbb{Z}^*_{3m} \times \{0\}$ and three elements from $\mathbb{Z}^*_{3m} \times \{1\}$.

(7) The triples in R can be paired, $\{d_{i1}, d_{i1}, d_{i3}\}$ and $\{d'_{i1}, d'_{i2}, d'_{i3}\}$, so that each pair contains three elements from $\mathbb{Z}^*_{3m} \times \{0\}$ and three elements from $\mathbb{Z}^*_{3m} \times \{1\}$.

A corresponding adder $A(S)$ consists of three sets:

$A_1 = (a_1, a_2, \ldots, a_{2m-2-6k}, f_1, f_2, \ldots, f_{6k})$ where $a_i, f_i \in \mathbb{Z}^*_{3m}$ and no element of $\mathbb{Z}^*_{3m}$ occurs more than once in A_1;

$A_2 = (g_{11}, g_{12}, \ldots, g_{k1}, g_{k2})$ where $g_{i1} \neq g_{i2}$ for all i A_1; $A_2 = (g_{11}, g_{12}, \ldots, g_{k1}, g_{k2})$ where $g_{i1} \neq g_{i2}$ for all i and $g_{ij} \in \{0, 1, 2\}$;

$A_3 = (h_{11}, h_{12}, \ldots, h_{k1}, h_{k2})$ where $h_{i1} \neq h_{i2}$ for all i and $h_{ij} \in \{0, 1, 2\}$.

$A(S)$ satisfies the following properties.

(1) $(S + A_1) \cup R = (\mathbb{Z}^*_{3m} \cup \{(\infty_i, 0), (\infty_i, 1) \mid i = 0, 1, \ldots, 3k-1\}) \times \{0, 1\}$.

(2) $\{c_{i1} + g_{i1}, c_{i2} + g_{i1}, c_{i3} + g_{i1}, c'_{i1} + g_{i2}, c'_{i2} + g_{i2}, c'_{i3} + g_{i2}\}$ contains precisely one representative from each congruence class $(x, 0)$ for x modulo 3 and one representative for each congruence class $(x, 1)$ for x modulo 3.

(3) $\{d_{i1} + h_{i1}, d_{i2} + h_{i1}, d_{i3} + h_{i1}, d'_{i1} + h_{i2}, d'_{i2} + h_{i2}, d'_{i3} + h_{i2}\}$ contains precisely one representative from each congruence class $(x, 0)$ for x modulo 3 and one representative for each congruence class $(x, 1)$ for x modulo 3.

THEOREM 4.11. *If there exists an intransitive starter* (S, R, C) *and a corresponding adder* $A(S)$ *for a* $(1, 1; 3)$*-frame over* $\mathbb{Z}_{3m} \times \{0, 1\}$*, then there is a* $(1, 1; 3)$*-frame of type* $6^m(6k)^1$.

Proof. As in the proof of Theorem 4.6, we use S and A_1 to construct a $3m \times 3m$ array M. We add $3k$ new rows and $3k$ new columns to this array.

Let B_i be a $3m \times 3$ array; label the rows $0, 1, \ldots, 3m-1$ and the columns $0, 2, 1$. Let $C_i = \{c_{i1}, c_{i2}, c_{i3}\}$ and let $C'_i = \{c'_{i1}, c'_{i2}, c'_{i3}\}$. In row j, column $(g_{i1} - j)$ modulo 3 of B_i, place the triple $C_i + j$, and in row j, column $(g_{i2} - j)$ modulo 3, place the triple C'_i for $j = 0, 1, \ldots, 3m-1$. Let $B = [B_1 B_2 \ldots B_k]$. B is a $3m \times 3k$ array. Similarly, we construct a $3k \times 3m$ array D from R and A_3. Let D_i be a $3 \times 3m$ array; label the columns $0, 1, \ldots, 3m-1$ and the rows $0, 2, 1$. Let $R_i = \{d_{i1}, d_{i2}, d_{i3}\}$ and let $R'_i = \{d'_{i1}, d'_{i2}, d'_{i3}\}$. In column j, row $(h_{i1} - j)$ modulo 3 of D_i, place the triple $R_i + j$, and in column j, row $(h_{i2} - j)$ modulo 3, place the triple R'_i for $j = 0, 1, \ldots, 3m-1$. Let $D = [D_1 D_2 \ldots D_k]^T$.

It is straightforward to verify that the following array constructed from M, B, and D gives us a $(1, 1; 3)$-frame of type $6^m(6k)^1$.

M	B
D	

■

LEMMA 4.12. *There exists a* $(1, 1; 3)$*-frame of type* 6^8.

Proof. Let $V = (\mathbb{Z}_{21} \cup \{\infty_0, \infty_1, \infty_2\}) \times \{0, 1\}$. The groups are

$$\{(i, 0), (i+7, 0), (i+14, 0), (i, 1), (i+7, 1), (i+14, 1)\}$$

Table 11. Frame of type 6^8.

Base Block (First Resolution)	Adder	Base Block + Adder (Orthogonal Resolution)
{(3, 0), (13, 0), (2, 1)}	0	
{(6, 0), (1, 1), (11, 1)}	2	
	0	{(13, 0), (17, 0), (18, 0)}
	1	{(4, 1), (5, 1), (9, 1)}
{(2, 0), (8, 0), (10, 0)}	17	{(19, 0), (4, 0), (6, 0)}
{(3, 1), (9, 1), (12, 1)}	10	{(13, 1), (19, 1), (1, 1)}
{(4, 0), (16, 0), (10, 1)}	6	{(10, 0), (1, 0), (16, 1)}
{(17, 0), (20, 0), (18, 1)}	9	{(5, 0), (8, 0), (6, 1)}
{(5, 0), (8, 1), (16, 1)}	15	{(20, 0), (2, 1), (10, 1)}
{(15, 0), (17, 1), (19, 1)}	1	{(16, 0), (18, 1), (20, 1)}
{(1, 0), (13, 1), (∞_0, 0)}	2	{(3, 0), (15, 1), (∞_0, 0)}
{(9, 0), (5, 1), (∞_1, 0)}	3	{(12, 0), (8, 1), (∞_1, 0)}
{(11, 0), (20, 1), (∞_2, 0)}	12	{(2, 0), (11, 1), (∞_2, 0)}
{(12, 0), (4, 1), (∞_0, 1)}	20	{(11, 0), (3, 1), (∞_0, 1)}
{(18, 0), (15, 1), (∞_1, 1)}	18	{(15, 0), (12, 1), (∞_1, 1)}
{(19, 0), (6, 1), (∞_2, 1)}	11	{(9, 0), (17, 1), (∞_2, 1)}

for $i = 0, 1, \ldots, 6$, and

$$\{(\infty_0, 0), (\infty_1, 0), (\infty_2, 0), (\infty_0, 1), (\infty_1, 1), (\infty_2, 1)\}.$$

The starter blocks and adders are as shown in Table 11. ■

LEMMA 4.13. *There exists a* $(1, 1; 3)$*-frame of type* 6^{10}.

Proof. Let $V = (\mathbb{Z}_{27} \cup \{\infty_0, \infty_1, \infty_2\}) \times \{0, 1\}$. The groups are

$$\{(i, 0), (i + 9, 0), (i + 18, 0), (i, 1), (i + 9, 1), (i + 18, 1)\}$$

for $i = 0, 1, \ldots, 8$, and

$$\{(\infty_0, 0), (\infty_1, 0), (\infty_2, 0), (\infty_0, 1), (\infty_1, 1), (\infty_2, 1)\}.$$

The starter blocks and adders are as shown in Table 12. ■

LEMMA 4.14. *There exists a* $(1, 1; 3)$*-frame of type* $6^{13}(12)^1$.

Proof. Let $V = (\mathbb{Z}_{39} \cup \{\infty_0, \infty_1, \infty_2, \infty_3, \infty_4, \infty_5\}) \times \{0, 1\}$. The groups are

$$\{(i, 0), (i + 13, 0), (i + 26, 0), (i, 1), (i + 13, 1), (i + 26, 1)\}$$

Table 12. Frame of type 6^{10}.

Base Block (First Resolution)	Adder	Base Block + Adder (Orthogonal Resolution)
$\{(3,0),(4,0),(11,0)\}$	0	
$\{(3,1),(4,1),(11,1)\}$	1	
	0	$\{(7,0),(11,0),(21,0)\}$
	1	$\{(7,1),(11,1),(21,1)\}$
$\{(1,0),(6,0),(12,0)\}$	2	$\{(3,0),(8,0),(14,0)\}$
$\{(2,0),(5,0),(1,1)\}$	11	$\{(13,0),(16,0),(12,1)\}$
$\{(7,0),(19,0),(2,1)\}$	13	$\{(20,0),(5,0),(15,1)\}$
$\{(8,0),(10,0),(16,1)\}$	16	$\{(24,0),(26,0),(5,1)\}$
$\{(13,0),(14,1),(26,1)\}$	15	$\{(1,0),(2,1),(14,1)\}$
$\{(17,0),(5,1),(21,1)\}$	5	$\{(22,0),(10,1),(26,1)\}$
$\{(20,0),(22,1),(25,1)\}$	22	$\{(15,0),(17,1),(20,1)\}$
$\{(21,0),(19,1),(24,1)\}$	4	$\{(25,0),(23,1),(1,1)\}$
$\{(22,0),(6,1),(12,1)\}$	7	$\{(2,0),(13,1),(19,1)\}$
$\{(26,0),(13,1),(15,1)\}$	20	$\{(19,0),(6,1),(8,1)\}$
$\{(14,0),(8,1),(\infty_0,0)\}$	17	$\{(4,0),(25,1),(\infty_0,0)\}$
$\{(15,0),(7,1),(\infty_1,0)\}$	24	$\{(12,0),(4,1),(\infty_1,0)\}$
$\{(16,0),(23,1),(\infty_2,0)\}$	1	$\{(17,0),(24,1),(\infty_2,0)\}$
$\{(23,0),(20,1),(\infty_0,1)\}$	10	$\{(6,0),(3,1),(\infty_0,1)\}$
$\{(24,0),(17,1),(\infty_1,1)\}$	26	$\{(23,0),(16,1),(\infty_1,1)\}$
$\{(25,0),(10,1),(\infty_2,1)\}$	12	$\{(10,0),(22,1),(\infty_2,1)\}$

for $i = 0, 1, \ldots, 12$, and

$$\{(\infty_i, 0), (\infty_i, 1) \mid i = 0, 1, \ldots, 5\}.$$

The starter blocks and adders are as shown in Table 13. ■

Combining the results in this section, we have the following result.

LEMMA 4.15. *There exist* $(1, 1; 3)$*-frames of types* $6^7, 6^8, 6^9, 6^{10}, 6^{13}, 6^{19}$, *and* $6^{13}(12)^1$.

5. Existence

In this section, we determine the spectrum of $KS_3(6x + 3; 1, 1)$ with at most 23 possible exceptions for x.

Our main recursive construction is the Group Divisible Design (GDD) Construction.

Table 13. Frame of type $6^{13}(12)^1$.

Base Block (First Resolution)	Adder	Base Block + Adder (Orthogonal Resolution)
{(3, 0), (1, 0), (8, 0)}	0	
{(6, 0), (7, 0), (17, 0)}	0	
{(3, 1), (1, 1), (8, 1)}	1	
{(6, 1), (7, 1), (17, 1)}	1	
	0	{(4, 0), (8, 0), (24, 0)}
	0	{(6, 0), (28, 0), (14, 0)}
	1	{(4, 1), (8, 1), (24, 1)}
	1	{(6, 1), (28, 1), (14, 1)}
{(2, 0), (4, 1), (10, 1)}	1	{(3, 0), (5, 1), (11, 1)}
{(4, 0), (2, 1), (5, 1)}	5	{(9, 0), (7, 1), (10, 1)}
{(5, 0), (9, 1), (21, 1)}	6	{(11, 0), (15, 1), (27, 1)}
{(9, 0), (12, 1), (27, 1)}	7	{(16, 0), (19, 1), (34, 1)}
{(10, 0), (15, 1), (24, 1)}	8	{(18, 0), (23, 1), (32, 1)}
{(11, 0), (18, 1), (36, 1)}	20	{(31, 0), (38, 1), (17, 1)}
{(14, 0), (38, 0), (33, 1)}	18	{(32, 0), (17, 0), (12, 1)}
{(16, 0), (34, 0), (25, 1)}	35	{(12, 0), (30, 0), (21, 1)}
{(18, 0), (30, 0), (29, 1)}	11	{(29, 0), (2, 0), (1, 1)}
{(22, 0), (31, 0), (16, 1)}	15	{(37, 0), (7, 0), (31, 1)}
{(23, 0), (29, 0), (35, 1)}	37	{(21, 0), (27, 0), (33, 1)}
{(24, 0), (27, 0), (20, 1)}	9	{(33, 0), (36, 0), (29, 1)}
{(12, 0), (32, 1), (∞_0, 0)}	10	{(22, 0), (3, 1), (∞_0, 0)}
{(15, 0), (38, 1), (∞_1, 0)}	19	{(34, 0), (18, 1), (∞_1, 0)}
{(19, 0), (34, 1), (∞_2, 0)}	21	{(1, 0), (16, 1), (∞_2, 0)}
{(20, 0), (37, 1), (∞_3, 0)}	24	{(5, 0), (22, 1), (∞_3, 0)}
{(21, 0), (31, 1), (∞_4, 0)}	4	{(25, 0), (35, 1), (∞_4, 0)}
{(25, 0), (14, 1), (∞_5, 0)}	34	{(20, 0), (9, 1), (∞_5, 0)}
{(28, 0), (11, 1), (∞_0, 1)}	30	{(19, 0), (2, 1), (∞_0, 1)}
{(32, 0), (22, 1), (∞_1, 1)}	3	{(35, 0), (25, 1), (∞_1, 1)}
{(33, 0), (30, 1), (∞_2, 1)}	29	{(23, 0), (20, 1), (∞_2, 1)}
{(35, 0), (23, 1), (∞_3, 1)}	14	{(10, 0), (37, 1), (∞_3, 1)}
{(36, 0), (28, 1), (∞_4, 1)}	2	{(38, 0), (30, 1), (∞_4, 1)}
{(37, 0), (19, 1), (∞_5, 1)}	17	{(15, 0), (36, 1), (∞_5, 1)}

THEOREM 5.1 (Group Divisible Design Construction). *Let G be a GDD*$(v; K; G_1, G_2, \ldots, G_m; 0, 1)$. *Suppose there exists a* $(1, 1; 3)$*-frame of type* 6^k *for each* $k \in K$ *and a* $KS_3(6|G_i|+3; 1, 1)$ *for each* i, $i = 1, 2, \ldots, m$. *Then there exists a* $KS_3(6v + 3; 1, 1)$.

Proof. We apply Theorem 4.3, the Fundamental Construction for frames, with $w(x) = 6$ for all x and then use Theorem 4.1, the Basic Frame Construction. ■

An immediate consequence of this and the existence results in the previous sections is the following construction using PBDs. This construction will be our main recursive construction.

THEOREM 5.2. *If there exists a* $PBD(v+1; \{7, 8, 9\})$*, then there exists a* $KS_3(6v+3; 1, 1)$.

Proof. We delete one element of a $PBD(v+1; \{7, 8, 9\})$ to construct a $\{7, 8, 9\}$-*GDD* with $|G_i| \in \{6, 7, 8\}$ for all i. Since there exist (1, 1; 3)-frames of type 6^k for $k = 7, 8, 9$ and $KS_3(6m+3; 1, 1)$ for $m = 6, 7, 8$, we apply Theorem 5.1 to construct a $KS_3(6v+3; 1, 1)$. ■

We use truncated transversal designs ([17]) to construct several other types of group divisible designs to use with Theorem 5.1.

LEMMA 5.3. *(Truncation of Transversal Designs)*

(1) *If there exists a* $TD(8, n)$*, then there exists a* $\{7, 8\}$-*GDD of type* $n^7 w^1$ *where* w *is an integer,* $0 \leq w \leq n$.

(2) *If there exists a* $TD(9, n)$*, then there exists a* $\{7, 8, 9\}$-*GDD of type* $n^7 w^1 y^1$ *where* w *and* y *are integers,* $0 \leq w, y \leq n$.

(3) *If there exists a* $TD(10, n)$*, then there exists a* $\{7, 8, 9, 10\}$-*GDD of type* $(n-1)^8 (w)^1 (y)^1$ *where* w *and* y *are integers,* $0 \leq w, y \leq n-1$.

(4) *There exists a* $\{7, 8, 9, w, 19\}$-*GDD of type* $7^{19-w} 8^w y^1$ *whenever* $0 \leq w \leq 19$ *and* $0 \leq y \leq 18$.

Proof.

(1) We truncate one group of a $TD(8, n)$ to size w.

(2) We truncate one group of a $TD(9, n)$ to size w and a second group to size y.

(3) We first delete one block of a $TD(10, n)$, then we truncate one group to size w and a second group to size y.

(4) A $TD(9, 19)$ is a $PBD(171; \{9, 19\})$. We first delete one point and use the truncated blocks to define groups of a $\{9, 19\}$-*GDD* of type $8^{19} 18^1$. Next, we delete $18 - y$ elements from the group of size 18 and $19 - w$ elements from a block of size 19. ■

We also need one special construction for a group divisible design on 131 elements.

LEMMA 5.4. *There exists a* $\{7, 8, 9\}$-*GDD of type* $8^{14}(12)^1(7)^1$.

Proof. Let D be a (120, 8, 1)-*RBIBD*; D has 17 resolution classes, $R_1, \ldots, R_{17}$. We first delete one element of D and denote the resulting resolution classes by $R'_1, \ldots, R'_{17}$. Next we adjoin a new element x_i to each block in R'_i for $i = 1, 2, \ldots, 12$. Finally, we consider the blocks of one of the remaining resolution classes, say R'_{17} as groups together with a new group $\{x_1, x_2, \ldots, x_{12}\}$. The resulting design is a $\{7, 8, 9\}$-GDD of type $8^{14}(12)^1(7)^1$ on 131 elements. ■

Finally we note that one other recursive construction is useful. It follows immediately from Theorem 4.2, the singular direct product for frames, and Theorem 4.1, the Basic Frame Construction. (See also [10, Theorem 4].)

THEOREM 5.5 (Direct Product Construction).

(1) If there exist a $(1, 1; 3)$*-frame of type* t^m*, three mutually orthogonal Latin squares of order n, and a* $KS_3(tn + 3; 1, 1)$*, then there exists a* $KS_3(tmn + 3; 1, 1)$.

(2) If there exist a $(1, 1; 3)$*-frame of type* t^m*, three mutually orthogonal Latin squares of order n, and a* $KS_3(tn + 1; 1, 1)$*, then there exists a* $KS_3(tmn + 1; 1, 1)$.

In order to apply Theorem 5.2, we first update the existence result for $PBD(v; \{7, 8, 9\})$s. Starting from the result in [33], we establish the following where the notation $[x, y]$ is used to denote the set of integers no smaller than x and no larger than y. Let $E_{789} = [10, 48] \cup [51, 55] \cup [59, 62]$ and $X_{789} = [93, 111] \cup [116, 118] \cup \{132\} \cup [138, 168] \cup [170, 174] \cup [180, 216] \cup [219, 223] \cup [228, 230] \cup [242, 258] \cup [261, 279] \cup [283, 286] \cup [298, 300] \cup [303, 307] \cup [311, 335] \cup [339, 342]$.

THEOREM 5.6. *For any integer* $v \geq 10$*, there is a* $PBD(v; \{7, 8, 9\})$ *except possibly when* v *is in* X_{789}*, and definitely when* v *is in* E_{789}.

Proof. This statement agrees with that in [33] except for seven values: 175, 176, 177, 178, 179, 259, and 260. For these values which were previously listed as possible exceptions, we use two known designs to give constructions. Abel [1] displays a $PBD(259; \{7\})$, which by construction has a parallel class. Adding a point, and placing it in each block of this parallel class gives a $PBD(260; \{7, 8\})$.

Janko and Tonchev [18] display a $PBD(175; \{7\})$ whose automorphism group has order 4200. Abel [1] gives a simpler presentation of the same design, over the elements $\mathbb{Z}_7 \times \mathbb{Z}_5 \times \mathbb{Z}_5$. The design has five base blocks, as follows:

$$B_1 = \{(0,0,0), (1,0,0), (2,0,0), (3,0,0), (4,0,0), (5,0,0), (6,0,0)\},$$
$$B_2 = \{(0,0,0), (1,1,3), (1,4,2), (2,2,2), (2,3,3), (4,2,0), (4,3,0)\},$$
$$B_3 = \{(0,0,0), (1,3,4), (1,2,1), (2,2,3), (2,3,2), (4,0,2), (4,0,3)\},$$
$$B_4 = \{(0,0,0), (1,1,2), (1,4,3), (2,1,1), (2,4,4), (4,0,1), (4,0,4)\},$$
$$B_5 = \{(0,0,0), (1,3,1), (1,2,4), (2,4,1), (2,1,4), (4,1,0), (4,4,0)\}.$$

Addition of each $(i, j, k) \in \mathbb{Z}_7 \times \mathbb{Z}_5 \times \mathbb{Z}_5$ to each of these base blocks yields a total of 725 blocks, 25 from the first and 175 each from the remainder. Using a program of Kreher and Stinson [20] to find cliques, all parallel classes were found. There are exactly 501, and of these the largest number of disjoint parallel classes is six. Hence the design is not resolvable. Nevertheless it can be used to produce further PBDs. Consider the four parallel classes in Table 14.

Table 14. Four parallel classes.

$B_2 + (0, 0, 0)$	$B_2 + (0, 2, 4)$	$B_2 + (1, 0, 1)$	$B_2 + (4, 0, 1)$	$B_2 + (5, 0, 0)$
$B_2 + (5, 2, 4)$	$B_2 + (6, 0, 1)$	$B_2 + (2, 0, 1)$	$B_2 + (3, 0, 0)$	$B_2 + (3, 2, 4)$
$B_2 + (0, 0, 1)$	$B_2 + (1, 0, 0)$	$B_2 + (1, 2, 4)$	$B_2 + (5, 0, 1)$	$B_2 + (6, 0, 0)$
$B_2 + (6, 2, 4)$	$B_2 + (3, 0, 1)$	$B_2 + (4, 0, 0)$	$B_2 + (4, 2, 4)$	$B_2 + (2, 0, 0)$
$B_2 + (2, 2, 4)$	$B_1 + (0, 0, 3)$	$B_1 + (0, 1, 2)$	$B_1 + (0, 1, 0)$	$B_1 + (0, 4, 0)$
$B_3 + (0, 0, 0)$	$B_3 + (0, 1, 2)$	$B_3 + (5, 0, 0)$	$B_3 + (5, 1, 2)$	$B_3 + (3, 0, 0)$
$B_3 + (3, 1, 2)$	$B_3 + (1, 0, 0)$	$B_3 + (1, 1, 2)$	$B_3 + (6, 0, 0)$	$B_3 + (6, 1, 2)$
$B_3 + (4, 0, 0)$	$B_3 + (4, 1, 2)$	$B_3 + (2, 0, 0)$	$B_3 + (2, 1, 2)$	$B_1 + (0, 0, 4)$
$B_1 + (0, 0, 1)$	$B_1 + (0, 2, 4)$	$B_1 + (0, 4, 3)$	$B_1 + (0, 3, 1)$	$B_1 + (0, 2, 2)$
$B_1 + (0, 2, 0)$	$B_1 + (0, 4, 2)$	$B_1 + (0, 1, 1)$	$B_1 + (0, 4, 0)$	$B_1 + (0, 1, 3)$
$B_5 + (0, 0, 0)$	$B_5 + (0, 1, 3)$	$B_5 + (1, 1, 1)$	$B_5 + (4, 1, 1)$	$B_5 + (5, 0, 0)$
$B_5 + (5, 1, 3)$	$B_5 + (6, 1, 1)$	$B_5 + (2, 1, 1)$	$B_5 + (3, 0, 0)$	$B_5 + (3, 1, 3)$
$B_5 + (0, 1, 1)$	$B_5 + (1, 0, 0)$	$B_5 + (1, 1, 3)$	$B_5 + (5, 1, 1)$	$B_5 + (6, 0, 0)$
$B_5 + (6, 1, 3)$	$B_5 + (3, 1, 1)$	$B_5 + (4, 0, 0)$	$B_5 + (4, 1, 3)$	$B_5 + (2, 0, 0)$
$B_5 + (2, 1, 3)$	$B_1 + (0, 4, 3)$	$B_1 + (0, 1, 2)$	$B_1 + (0, 3, 4)$	$B_1 + (0, 3, 3)$
$B_4 + (0, 0, 0)$	$B_4 + (1, 2, 0)$	$B_4 + (1, 4, 1)$	$B_4 + (4, 2, 0)$	$B_4 + (4, 4, 1)$
$B_4 + (5, 0, 0)$	$B_4 + (6, 2, 0)$	$B_4 + (6, 4, 1)$	$B_4 + (2, 2, 0)$	$B_4 + (2, 4, 1)$
$B_4 + (3, 0, 0)$	$B_4 + (0, 2, 0)$	$B_4 + (0, 4, 1)$	$B_4 + (1, 0, 0)$	$B_4 + (5, 2, 0)$
$B_4 + (5, 4, 1)$	$B_4 + (6, 0, 0)$	$B_4 + (3, 2, 0)$	$B_4 + (3, 4, 1)$	$B_4 + (4, 0, 0)$
$B_4 + (2, 0, 0)$	$B_1 + (0, 2, 3)$	$B_1 + (0, 2, 2)$	$B_1 + (0, 1, 0)$	$B_1 + (0, 3, 3)$

It is easy to verify that each pair of these parallel classes intersects in exactly one block, and that no three of the parallel classes share a block. Hence we can add four new points, each one to all blocks of one of the parallel classes, to obtain a $PBD(179; \{7, 8, 9\})$. Deleting 0, 1, 2, 3, or 4 of these new points yields a $PBD(v, \{7, 8, 9\})$ for $v \in \{179, 178, 177, 176, 175\}$. ■

We are now in a position to determine the spectrum of $KS_3(6x + 3; 1, 1)$ with at most 23 possible exceptions for x. We first take care of the cases for $4 \leq x \leq 61$. Let $N = \{3, 9, 11, 15, 16, 17, 19, 23, 24, 25, 27, 29, 30, 31, 33, 38, 39, 41, 43, 44, 47, 58, 59\}$.

LEMMA 5.7. *There exists a* $KS_3(6x + 3; 1, 1)$ *for* $4 \leq x \leq 61$ *except possibly for* $x \in N$.

Proof. We noted the existence of $KS_3(6x + 3; 1, 1)$ for $x = 4, 6, 8, 10$, and 13 in Section 2; $x = 40$ is also done using Theorem 2.2 with $q = 3$ and $n = 5$. For $x \in \{5, 7, 12, 14, 18, 20, 21, 22, 26, 34, 36, 42, 46, 50, 52\}$, $KS_3(6x + 3; 1, 1)$ are constructed directly in Section 3. For $x = 28, 32, 35, 37, 45, 51, 53$, we apply the direct product construction, Theorem 5.5 to construct $KS_3(6x + 3; 1, 1)$. We use Theorem 5.5(1) with a frame of type 6^7 and $n \in \{4, 5\}$ for $x \in \{28, 35\}$, with a frame of type 6^8 and $n = 4$ for $x = 32$, and with a frame of type 6^9 and $n = 5$ for $x = 45$. We use Theorem 5.5(2) with a frame of type 4^7 and $n \in \{8, 11\}$ for $x \in \{37, 51\}$, and with a frame of type 4^{10} and $n = 8$ for $x = 53$. (We recall that the frames of type 4^7 and 4^{10} exist by Lemma 4.5.)

Next, we apply Theorem 5.2 to take care of $x = 48, 49, 55, 56$, and 57. The required PBDS exist by Theorem 5.6. Finally, we use Lemma 5.3 and Theorem 5.1 for the remaining three cases, $x = 54, 60, 61$. For $x = 54$, we use Lemma 5.3(1) with $n = 7$ and $w = 5$. For $x = 60$ and $x = 61$, we use Lemma 5.3(2) with $n = 8$, $w = 0$, and $y = 4, 5$. ■

Next we consider the spectrum for $62 \leq x \leq 341$.

LEMMA 5.8. *There exists a* $KS_3(6x+3; 1, 1)$ *for* $62 \leq x \leq 341$.

Proof. We use the existence of $PBD(x+1; \{7, 8, 9\})$ and Theorem 5.2 for $x \in [62, 91] \cup [111, 114] \cup [118, 130] \cup [132, 136] \cup \{168\} \cup [174, 178] \cup [216, 217] \cup [223, 226] \cup [230, 240] \cup [258, 259] \cup [279, 281] \cup [286, 296] \cup [300, 301] \cup [307, 309] \cup [335, 337]$.

All of the remaining cases except $x = 131$, 173, 179 and 227 can be done using Lemma 5.3 and Theorem 5.1. Table 15 contains a list of the parameters needed for each application of Lemma 5.3.

The four remaining cases are done as follows. We use Lemma 5.4 and Theorem 5.1 to take care of $x = 131$.

The cases $x = 173$ and $x = 179$ are done using Theorem 4.3, the Fundamental Construction for frames, and then Theorem 4.1. We start with a $TD(14, 13)$ and give weight 6 to each point in 13 of the groups and assign two or five points weight 12 and the remaining eleven or eight weight 0 in the last group. We then use the frames of types 6^{13} and $6^{13}12^1$ to produce frames of type $78^{13}24^1$ and $78^{13}60^1$ and apply Theorem 4.1 with $u = 3$.

Table 15. Applications of Lemma 5.3.

x	Case	n	w	y	x	Case	n	w	y
54	(1)	7	5		92–94	(3)	11	8	4–6
95–99	(2)	13	0	4–8	100–103	(2)	13	4	5–8
104–107	(2)	13	8	5–8	108–110	(2)	13	12	5–7
115–116	(2)	13	12	12–13	117	(2)	13	13	13
137–141	(4)	19	0	4–8	142	(4)	19	1	8
143	(4)	19	0	10	144–148	(4)	19	7	4–8
149–151	(4)	19	10	6–8	152–154	(4)	19	13	6–8
155–157	(4)	19	10	12–14	158–160	(4)	19	13	12–14
161	(4)	19	10	18	162–166	(3)	19	14	4–8
167	(3)	19	10	13	169–170	(3)	19	12	13–14
171–172	(3)	19	14	13–14	180–184	(3)	23	0	4–8
185–189	(3)	23	5	4–8	190–194	(3)	23	10	4–8
195–198	(3)	23	14	5–8	199–202	(3)	23	18	5–8
203–206	(3)	23	22	5–8	207–208	(3)	23	18	13–14
209–210	(3)	23	20	13–14	211	(3)	23	22	13
212–215	(3)	27	0	4–7	218–222	(3)	27	6	4–8
228–229	(2)	32	0	4–5	241–245	(2)	32	13	4–8
246–250	(2)	32	18	4–8	251–254	(2)	32	22	5–8
255–257	(2)	32	26	5–7	260–264	(2)	32	32	4–8
265–267	(2)	37	0	6–8	268–272	(2)	37	5	4–8
273–277	(2)	37	10	4–8	278	(2)	37	14	5
282–285	(2)	37	18	5–8	297–299	(2)	40	12	5–7
302–306	(2)	40	18	4–8	310–314	(2)	40	26	4–8
315	(2)	40	28	7	316–320	(2)	40	32	4–8
321–325	(2)	40	37	4–8	326–328	(2)	40	40	6–8
329–331	(2)	40	37	12–14	332–334	(2)	40	40	12–14
338–339	(2)	40	22	36–37	340–341	(2)	40	26	34–35

Finally, we use the direct product construction for $x = 227$. We apply Theorem 5.5 Case (2) with $t = 2$, $m = 22$, and $n = 31$. (See Lemma 4.4 for the frame of type 2^{22}.) ■

We combine these results to prove our main result.

THEOREM 5.9. *Let $v = 6x+3$ where x is a non-negative integer. There exists a $KS_3(v; 1, 1)$ for $x = 0$ and $x \geq 3$ except possibly for $x \in N$ where*

$$N = \{3, 9, 11, 15, 16, 17, 19, 23, 24, 25, 27, 29, 30, 31, 33, 38, 39, 41, 43, 44, 47, 58, 59\}.$$

Furthermore, there do not exist $KS_3(v; 1, 1)$ for $v = 9$ and $v = 15$.

Proof. The nonexistence of $KS_3(6x + 3; 1, 1)$ for $x \in \{1, 2\}$ is established in [36]. By Lemmas 5.7 and 5.8, there exist $KS_3(6x + 3; 1, 1)$ for $3 \leq x \leq 341$ except possibly for $x \in N$. Since there exist $PBD(x + 1; \{7, 8, 9\})$ for $x \geq 342$, we can apply Theorem 5.2 to construct $KS_3(6x + 3; 1, 1)$ for $x \geq 342$. ■

Acknowledgments

Research of the authors is supported by ARO grant DAAG55-98-1-0272 (Colbourn), DAAD19-01-1-0406 (Colbourn and Ling), and NSF grant DMS9753244 (Lamken). We also thank Scott Vanstone for his many suggestions and discussions on the existence of $KS_3(v; 1, 1)$.

References

1. R. Julian R. Abel, Some new BIBDs with block size 7, *J. Combinatorial Designs,* Vol. 8 (2000) pp. 146–150.
2. R. Julian R. Abel, A. E. Brouwer, C. J. Colbourn and J. H. Dinitz, Mutually orthogonal Latin squares (MOLS), In (C. J. Colbourn, J. H. Dinitz, eds.), *The CRC Handbook of Combinatorial Designs,* CRC Press, Boca Raton, FL (1996) pp. 111–142.
3. R. Julian R. Abel and S. C. Furino, Resolvable and near resolvable designs, In (C. J. Colbourn, J. H. Dinitz, eds.), *The CRC Handbook of Combinatorial Designs,* CRC Press, Boca Raton, FL (1996) pp. 87–94.
4. I. Anderson, *Combinatorial Designs, Construction Methods,* Ellis Horwood, Limited, Chichester, England (1990).
5. R. D. Baker, Orthogonal line packings of $PG_{2m-1}(2)$, *J. Combinat. Theory (A)*, Vol. 36 (1984) pp. 245–248.
6. F. E. Bennett, H-D. O. F. Gronau, A. C. H. Ling and R. C. Mullin, PBD closure, In (C. J. Colbourn, J. H. Dinitz, eds.), *The CRC Handbook of Combinatorial Designs,* CRC Press, Boca Raton, FL (1996) pp. 203–213.
7. T. Beth, D. Jungnickel and H. Lenz, *Design Theory*, Cambridge University Press (1986).
8. C. J. Colbourn and A. Rosa, *Triple Systems,* Oxford University Press, Oxford, UK (1999).
9. C. J. Colbourn and S. A. Vanstone, Doubly resolvable twofold triple systems, In *Proc. Eleventh Manitoba Conf. Numer. Math. Computing, Congressus Numerantium,* Vol. 34 (1982) pp. 219–223.
10. C. J. Colbourn, D. Curran and S. A. Vanstone, Recursive constructions for Kirkman squares with block size 3, *Utilitas Math.,* Vol. 32 (1987) pp. 169–174.
11. D. G. Curran and S. A. Vanstone, Doubly resolvable designs from generalized Bhaskar Rao designs, *Discrete Mathematics,* Vol. 73 (1988) pp. 49–63.

12. J. H. Dinitz and D. R. Stinson, Room squares and related designs, In *Contemporary Design Theory: A Collection of Surveys,* Wiley, New York (1992) pp. 137–204.
13. R. Fuji-Hara and S. A. Vanstone, On the spectrum of doubly resolvable designs, *Congressus Numer* (1980) pp. 399–407.
14. R. Fuji-Hara and S. A. Vanstone, The existence of orthogonal resolutions of lines in $AG(n, q)$, *J. Combinatorial Theory (A),* Vol. 45 (1987) pp. 139–147.
15. R. Fuji-Hara and S. A. Vanstone, Transversal designs and doubly resolvable designs, *Europ. J. Combin.,* Vol. 1 (1980) pp. 219–223.
16. S. C. Furino, Y. Miao and J. Yin, *Frames and Resolvable Designs,* CRC Press, Boca Raton, FL (1996).
17. H. Hanani, Balanced incomplete block designs and related designs, *Discrete Math.* (1975) pp. 253–369.
18. Z. Janko and V. D. Tonchev, New designs with block size 7, *J. Combin. Theory (A),* Vol. 83 (1998) pp. 152–157.
19. T. P. Kirkman, Query VI, *Lady's and Gentleman Diary* (1850) p. 48.
20. D. L. Kreher and D. R. Stinson, *Combinatorial Algorithms: Generation, Enumeration and Search*, CRC Press, Boca Raton, FL (1998).
21. E. R. Lamken, 3-complementary frames and doubly near resolvable $(v, 3, 2)$-*BIBD*s, *Discrete Math.,* Vol. 88 (1991) pp. 59–78.
22. E. R. Lamken, Constructions for generalized balanced tournament designs, *Discrete Mathematics,* Vol. 131 (1994) pp. 127–151.
23. E. R. Lamken, Constructions for resolvable and near resolvable (v, k, λ)-*BIBD*s, In (D. K. Ray-Chaudhuri, ed.) *Coding Theory and Design Theory, Part II: Design Theory,* Springer-Verlag, New York (1990) pp. 236–250.
24. E. R. Lamken, Coverings, Orthogonally Resolvable Designs and Related Combinatorial Configurations, Ph.D. Thesis, University of Michigan (1983).
25. E. R. Lamken, *Designs with orthogonal resolutions and decompositions of edge-colored complete graphs*, in preparation.
26. E. R. Lamken, The existence of doubly resolvable $(v, 3, 2)$-*BIBD*s, *J. Combin. Theory (A),* Vol. 72 (1995) pp. 50–76.
27. E. R. Lamken, The existence of $KS_3(v; 2, 4)$s, *Discrete Math.,* Vol. 186 (1998) pp. 195–216.
28. E. R. Lamken, Generalized balanced tournament designs, *Transactions of the AMS,* Vol. 318 (1990) pp. 473–490.
29. E. R. Lamken and S. A. Vanstone, Existence results for doubly near resolvable $(v, 3, 2)$-*BIBD*s, *Discrete Math.,* Vol. 120 (1993) pp. 135–148.
30. E. R. Lamken and S. A. Vanstone, The existence of $KS_k(v; \mu, \lambda)$: I. the main constructions, *Utilitas Math.,* Vol. 27 (1985) pp. 111–130.
31. E. R. Lamken and S. A. Vanstone, The existence of $KS_k(v; \mu, \lambda)$: II. special constructions, *Utilitas Math.,* Vol. 27 (1985) pp. 131–155.
32. E. R. Lamken and S. A. Vanstone, On a class of Kirkman squares of index 2, *J. of Australian Math. Society (A),* Vol. 44 (1988) pp. 33–41.
33. A. C. H. Ling, X. J. Zhu, C. J. Colbourn and R. C. Mullin, Pairwise balanced designs with consecutive block sizes, *Des. Codes Crypt.,* Vol. 10 (1997) pp. 203–222.
34. J. Lu, An existence theory for resolvable balanced incomplete block designs, *Acta Math. Sinica,* Vol. 27 (1984) pp. 458–468 (in Chinese).
35. R. Mathon and S. A. Vanstone, Doubly resolvable Kirkman systems, *Congressus Num.,* Vol. 29 (1980) pp. 611–125.
36. R. Mathon and S. A. Vanstone, On the existence of doubly resolvable Kirkman systems and equidistant permutation arrays, *Discrete Math.,* Vol. 30 (1980) pp. 157–172.
37. R. C. Mullin and W. D. Wallis, The existence of Room squares, *Aequationes Math.,* Vol. 1 (1975) pp. 1–7.
38. A. V. Nazarok, Orthogonal starters and multidimensional Kirkman hypercubes, *Discrete Math. Appl.,* Vol. 5 (1996) pp. 541–545.
39. A. V. Nazarok, On the existence of multidimensional Kirkman hypercubes of a certain class, (Ukranian), *Dopov./Dokl. Akad. Nauk Ukraini,* Vol. 7 (1993) pp. 34–35.

40. A. Rosa and S. A. Vanstone, Starter-adder techniques for Kirkman squares and Kirkman cubes of small sides, *Ars Combinatoria,* Vol. 14 (1982) pp. 199–212.
41. A. Rosa and S. A. Vanstone, On the existence of strong Kirkman cubes of order 39 and block size 3, *Ann. of Discrete Math.,* Vol. 26 (1983) pp. 309–320.
42. D. R. Stinson and S. A. Vanstone, A Kirkman square of order 51 and block size 3, *Discrete Mathematics,* Vol. 35 (1985) pp. 107–111.
43. V. D. Tonchev and S. A. Vanstone, On Kirkman triple systems of order 33, *Discrete Math.,* Vol. 106/107 (1992) pp. 493–496.
44. S. A. Vanstone, Doubly resolvable designs, *Discrete Mathematics,* Vol. 29 (1980) pp. 77–86.
45. S. A. Vanstone, On mutually orthogonal resolutions and near resolutions, *Ann. of Discrete Mathematics,* Vol. 15 (1982) pp. 357–369.
46. R. M. Wilson, Constructions and uses of pairwise balanced designs, In (M. Hall, Jr and J. H. van Lint, eds.), *Proc. NATO Advanced Study Institute in Combinatorics,* Nijenrode Castle, Breukelen (1974) pp. 19–42.