Available at
www.ElsevierMathematics.com
POWERED BY SCIENCE DIRECT
Discrete Mathematics 274 (2004) 199–212
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Embedding Steiner triple systems into Steiner systems $S(2,4,v)$

Mariusz Meszka[a], Alexander Rosa[b]

[a]Department of Applied Mathematics, AGH Krakow, Krakow, Poland
[b]Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario, Canada L8S 4K1

Received 26 March 2002; received in revised form 13 November 2002; accepted 3 December 2002

Abstract

We initiate a systematic study of embeddings of Steiner triple systems into Steiner systems $S(2,4,v)$. We settle the existence of an embedding of the unique STS(7) and, with one possible exception, of the unique STS(9) into $S(2,4,v)$. We also obtain bounds for embedding sizes of Steiner triple systems of larger orders.

Keywords: Steiner triple systems; Steiner systems $S(2,4,v)$; Embedding

1. Introduction

A *Steiner system* $S(t,k,v)$ is a pair $(V,\mathscr{B})$ where V is a v-set, $v>0$, and $\mathscr{B}$ is a collection of k-subsets of V called *blocks* such that each t-subset of V is contained in exactly one block. A Steiner system $S(2,3,v)$ is called a *Steiner triple system* of *order* v, briefly STS(v). If we replace in the above definition the words "exactly one" with "at most one", we obtain the definition of a *partial Steiner system* $S(t,k,v)$.

It is well known that an STS(v) exists if and only if $v \equiv 1,3 \pmod 6$ [3], and that an $S(2,4,w)$ exists if and only if $w \equiv 1,4 \pmod{12}$. We refer to these orders as *admissible*.

A *parallel class* in an STS(v) $(V,\mathscr{B})$ is a set of blocks which partition the set V. An STS(v) $(V,\mathscr{B})$ is *resolvable* if its set of blocks $\mathscr{B}$ can be partitioned into parallel classes. Such a partition is termed *resolution*. A *Kirkman triple system* KTS(v) of

E-mail address: rosa@mcmail.cis.mcmaster.ca (A. Rosa).

0012-365X/03/$ - see front matter
doi:10.1016/S0012-365X(03)00089-X

order v is an STS(v) together with a particular resolution R. It is well known that a resolvable STS(v) [and thus a KTS(v)] exists if and only if $v \equiv 3 \pmod 6$ [3].

A Steiner system $S(t,k,v)$ $(V,\mathscr{B})$ is *embedded* in a Steiner system $S(t',k',w)$ $(W,\mathscr{C})$ if $V \subset W$, and $\mathscr{C}|V = \mathscr{B}$, i.e. $\mathscr{B} = \{B : B \in \mathscr{C}, B \subset V\}$. In this case we also say that $(W,\mathscr{C})$ *contains* $(V,\mathscr{B})$ *as a subsystem*.

The best studied examples of embeddings of Steiner systems are those when $t=t'=2$ or 3 and $k=k'=3$ or 4. The following two well-known theorems provide a definite answer in two of the cases.

The Doyen–Wilson Theorem (Doyen and Wilson [5]). *An* STS(v) *can be embedded in an* STS(w) *if and only if* $v=w$ *or* $w \geqslant 2v+1$, *and both* v,w *are admissible*.

The Rees–Stinson Theorem (The Rees and Stinson [16]). *A Steiner system* $S(2,4,v)$ *can be embedded in a Steiner system* $S(2,4,w)$ *if and only if* $v=w$ *or* $w \geqslant 3v+1$ *and both* v,w *are admissible*.

Hartman [7] has made substantial progress towards proving the conjecture that a Steiner system $S(3,4,v)$ (also called *Steiner quadruple system*) can be embedded into a Steiner system $S(3,4,w)$ if and only if $w \geqslant 2v$ and $v,w \equiv 2,4 \pmod 6$ [or $v=w$, of course]. However, this conjecture remains open.

Much less appears known in the case when $t=t'$ but $k<k'$. Of course, there are the well-known geometric examples obtained from embedding affine planes in projective planes, or, in our notation, embedding Steiner systems $S(2,q,q^2)$ into Steiner systems $S(2,q+1,q^2+q+1)$, and, more generally, embedding affine spaces into projective spaces. Other examples include examination of projective embeddings of small Steiner triple systems by Limbos [11], and embeddings of affine and projective spaces in projective planes [8].

In this paper we concentrate on the case when $t=t'=2, k=3, k'=4$, i.e. on the question of embeddings of Steiner triple systems into Steiner systems $S(2,4,v)$. To the best of our knowledge, this case has not been studied systematically, although sporadic results concerning this case can be found in the sources cited already, and also in [17], and [9]; in the latter article, one finds examples of STS(7) embedded in $S(2,4,25)$.

To see that the general question "Which Steiner triple systems can be embedded in a Steiner system $S(2,4,v)$?" is not easy and is not likely to be settled in one fell swoop, it suffices to realize that this question includes, as a special case, the question of de Resmini [17] about the existence of the "century design", as well as some additional unsolved problems related to specialized colourings of Steiner systems $S(2,4,v)$ with blocks having prescribed colour patterns [13].

There is another feature which distinguishes this problem from embedding problems when $t=t', k=k'$: there is no *replacement property*. If we have an embedding of, say, an STS(v) into an STS(w), then *any other* STS(v) can also be embedded in an STS(w). This is no longer true when one considers embeddings where $k<k'$, e.g. those of STSs into $S(2,4,v)$.

In this paper, we settle the existence of an embedding of the unique STS(7) (i.e. the projective plane of order 2, or Fano plane) and, with one possible exception, of the unique STS(9) (i.e. the affine plane of order 3) into Steiner systems $S(2,4,w)$: an $S(2,4,w)$ containing STS(7) exists if and only if $w \geqslant 25, w \equiv 1,4 \pmod{12}$, while an $S(2,4,w)$ containing STS(9) exists if and only if $w=13$ or $w \geqslant 28$, $w \equiv 1,4 \pmod{12}$, except possibly when $w=37$. We also obtain bounds for embedding sizes of Steiner triple systems of larger orders.

2. Preliminaries and necessary conditions

In addition to the Rees–Stinson Theorem given in the Introduction, we will also make use of the following theorems.

Ganter's Theorem (Ganter [6]). *Every partial Steiner system $S(2,k,v)$ can be embedded in some Steiner system $S(2,k,w)$.*

However, we must note that for $k>3$ the order w of the containing Steiner system is, in general, exponential in v.

Theorem 1 (Mullin et al. [14] and Colbourn and Rosa [3]). *A resolvable* STS$(2v+1)$ *containing an* STS(v) *exists when* $v \equiv 1 \pmod 6$, *except possibly if* $v \in \{115, 145, 205, 265, 355, 415, 649, 697\}$.

Theorem 2 (Rees and Stinson [15]). *A* KTS(v) *can be embedded in a* KTS(w) *if and only if $w=v$ or $w \geqslant 3v$, $w \equiv v \equiv 3 \pmod 6$.*

Let $(V,\mathscr{B})$ be an STS(v). Define $E(V,\mathscr{B})$, the *embedding spectrum* for $(V,\mathscr{B})$, as follows:

$E(V,\mathscr{B}) = \{w$: there exists an $S(2,4,w)$ containing $(V,\mathscr{B})$ as a subsystem$\}$.

Then $E(v)$, *the embedding spectrum for* v, is defined as

$E(v) = \bigcup E(V,\mathscr{B})$,

where the union is taken over all STS(v) $(V,\mathscr{B})$.

For an STS(v) $(V,\mathscr{B})$, define further

$m(V,\mathscr{B}) = \min E(V,\mathscr{B})$, $m(v) = \min E(v)$, and

$q(V,\mathscr{B}) = \min\{q$: there exists an $S(2,4,w)$ containing $(V,\mathscr{B})$ as a subsystem for all admissible $w \geqslant q\}$.

The quantity $q(v)$ is defined similarly:

$q(v) = \min\{q$: there exists an $S(2,4,w)$ containing some STS(v) as a subsystem for all admissible $w \geqslant q\}$.

The following is an easy consequence of Ganter's Theorem.

Theorem 3. *Every Steiner triple system $S=(V,\mathscr{B})$ can be embedded in some $S(2,4,w)$.*

Proof. It suffices to convert the STS S into a partial Steiner system $S(2,4,v)$. This can be done in many ways, one of which is the following. Let $C=\{c_1,\ldots,c_t\}$ be the set of block colour classes (a block colour class is the set of blocks coloured with the same colour) in any block colouring of S. Then form a partial Steiner system $S(2,4,v)$ on the set $V \cup C$ by extending each triple T of the colour class c_i to a 4-subset $T \cup \{c_i\}$. The rest follows from Ganter's Theorem. □

Thus the sets $E(V,\mathscr{B})$ are nonempty for all STS $(V,\mathscr{B})$, and so are, of course, the sets $E(v)$ for all admissible v. Furthermore, $m(V,\mathscr{B})$, $q(V,\mathscr{B})$, $m(v)$ and $q(v)$ are all well defined.

Lemma 4. *For all* $v \equiv 1,3 \pmod 6$, $q(v) \leqslant 3m(v)+1$.

Proof. This follows directly from the Rees–Stinson Theorem. □

Lemma 5. *For all* STS(v) $(V,\mathscr{B})$, $m(V,\mathscr{B}) \geqslant (3v-1)/2$.

Proof. Suppose an STS(v) $(V,\mathscr{B})$ is embedded as a subsystem in an $S(2,4,w)$ $(W,\mathscr{C})$. Every element of V appears $r=(v-1)/2$ times in $\mathscr{B}$, and every element of W must appear $r'=(w-1)/3$ times in $\mathscr{C}$. We must have $r' \geqslant r$, and the Lemma follows. □

Theorem 6. *For an* STS(v) $(V,\mathscr{B})$, *we have* $m(V,\mathscr{B})=(3v-1)/2$ *if and only if* $v \equiv 3$ *or* $9 \pmod{24}$, *and* $(V,\mathscr{B})$ *is resolvable*.

Proof. A well known construction for Steiner systems $S(2,4,v)$ (the $3v+1$ construction, see, e.g. [13]) embeds an $S(2,4,v)$ into an $S(2,4,3v+1)$ with the help of a KTS$(2v+1)$: all blocks of the containing $S(2,4,3v+1)$ *not* in the subsystem $S(2,4,v)$ are obtained by taking the parallel classes $R_1,\dots,R_v$ of the KTS$(2v+1)$, and adding to each block of the parallel class R_i the ith element of the subsystem $S(2,4,v)$. At the same time, this construction provides an embedding of the KTS$(2v+1)$ into an $S(2,4,3v+1)$. Since $v \equiv 1,4 \pmod{12}$, we have $2v+1 \equiv 3,9 \pmod{24}$. The necessity follows from the fact that when $2v+1 \not\equiv 3,9 \pmod{24}$, $3v+1$ is not an order of an $S(2,4,v)$. □

Corollary 7. *For all* $v \equiv 1,3 \pmod 6$, $m(v) \geqslant (3v-1)/2$.

Our next theorem provides linear bounds for the quantity $m(v)$. However, we first need the following lemma.

Lemma 8. *For every* $v \equiv 1 \pmod 6$, *except possibly for* $v=13$, *there exists an* STS(v) *which can be embedded in a resolvable* STS$(3v)$.

Proof. A resolvable STS(21) containing an STS(7) is given in [12]. Let now $v \equiv 1 \pmod 6$, $v \geqslant 19$. For each such v, there exists a *Hanani triple system* of order v, i.e. an STS(v) whose set of triples can be partitioned into $(v-1)/2$ almost parallel classes containing $(v-1)/3$ triples each, and one 'short' partial parallel class of $(v-1)/6$ triples. Let $(V,\mathscr{B})$ be a Hanani triple system of order v, R_0 be its 'short' parallel class, and $R_1,\dots,R_{(v-1)/2}$ its almost parallel classes. Put now $W=V\times\{1,2,3\}$, and put a copy of $(V,\mathscr{B})$ on each of the sets $V\times\{j\}$, $j=1,2,3$. For each $i=1,\dots,(v-1)/2$ the set

$S_i=\bigcup_{j=1}^{3} R_{ij}\cup\{\{x_1^i,x_2^i,x_3^i\}\}$ is a parallel class of triples on W where w.l.o.g. x^i is the isolated point of the almost parallel class R_i. In addition, let $S_0=\bigcup_{j=1}^{3} R_{0j}\cup\{\{x_1^k,x_2^k,x_3^k\}$: $k\in V\setminus\{1,\dots,(v-1)/2\}$; S_0 is another parallel class of triples on W. Let now $Q_1,\dots,Q_v$ be the parallel classes of a resolvable transversal design TD$(3,v)$; such a transversal design is equivalent to a pair of orthogonal latin squares of order v, and clearly exists. W.l.o.g., we may assume that $Q_1=\{\{x_1^i,x_2^i,x_3^i\}: i\in V\}$. But then the parallel classes $S_0,S_1,\dots,S_{(v-1)/2},Q_2,\dots,Q_v$ are the $(v+1)/2+v-1=(3v-1)/2$ parallel classes of a resolvable STS$(3v)$. □

Theorem 9. *Let* $v\equiv 1,3 \pmod 6$. *Then*

(i) $m(v)=(3v-1)/2$ if $v\equiv 3,9 \pmod{24}$,
(ii) $m(v)\leqslant 3v+1$ *if* $v\equiv 1 \pmod{12}$, *except possibly when* $v\in X_0=\{145,205,265,649,697\}$,
(iii) $m(v)\leqslant (9v+17)/2$ *if* $v\equiv 15 \pmod{24}$,
(iv) $m(v)\leqslant (9v+35)/2$ *if* $v\equiv 21 \pmod{24}$,
(v) $m(v)\leqslant (9v-1)/2$ *if* $v\equiv 19 \pmod{24}$, and
(vi) $m(v)\leqslant 9v+13$ *if* $v\equiv 7 \pmod{24}$, *except possibly when* $v=415$.

Proof. (i) Follows from Theorem 6. For (ii), let $v=12t+1$. Take a KTS$(2v+1=24t+3)$ containing a sub-STS$(v=12t+1)$; such a KTS always exists, except possibly when $v\in X_0$ [3]. By Theorem 6, this KTS can be embedded in an $S(2,4,3v+1=36t+4)$. For (iii), let $v=24t+15$. Embed a KTS$(24t+15)$ in a KTS$(3v+6=72t+51)$ (which is possible by Theorem 2); the latter can be embedded into an $S(2,4,108t+76=(9v+17)/2)$. For (iv), let $v=24t+21$. Embed now a KTS$(24t+21)$ into a KTS$(3v+12=72t+75)$ (which is again possible by Theorem 2); the latter can be embedded in an $S(2,4,108t+112=(9v+35)/2)$. For (v), use Lemma 8 to embed an STS$(24t+19)$ in a KTS$(3v=72t+57)$. By Theorem 6, this KTS can be embedded in an $S(2,4,108t+85=(9v-1)/2)$. Finally, for (vi), let $v=24t+7$. Take a KTS$(2v+1=48t+15)$ with a sub-STS$(v=24t+7)$; such a KTS always exists, except possibly when $v=415$ [3]. By Theorem 2, this KTS can be embedded in a KTS$(6v+9=144t+51)$ which in turn can be embedded in an $S(2,4,9v+13=216t+76)$. □

Corollary 10. *Let* $v\equiv 1,3 \pmod 6$. *Then*

(i) $q(v)\leqslant (9v-1)/2$ *if* $v\equiv 3,9 \pmod{24}$,
(ii) $q(v)\leqslant 9v+4$ *if* $v\equiv 1 \pmod{12}$ *except possibly when* $v\in X_0$,
(iii) $q(v)\leqslant (27v+53)/2$ *if* $v\equiv 15 \pmod{24}$,
(iv) $q(v)\leqslant (27v+107)/2$ *if* $v\equiv 21 \pmod{24}$,
(v) $q(v)\leqslant (27v-1)/2$ *if* $v\equiv 19 \pmod{24}$, *and*
(vi) $q(v)\leqslant 27v+40$ *if* $v\equiv 7 \pmod{24}$ *except possibly when* $v=415$.

Proof. This follows directly from Lemma 4 and Theorem 9. □

The next theorem shows that the bounds of Theorem 9 can be improved if an additional condition is satisfied.

Theorem 11. (i) *Let* $v \equiv 15 \pmod{24}$, *and suppose that there exists a* KTS$(2v+3)$ *containing an* STS(v) *as a subsystem. Then* $m(v) \leqslant 3v+4$.

(ii) *Let* $v \equiv 21 \pmod{24}$, *and suppose that there exists a* KTS$(2v+9)$ *containing an* STS(v) *as a subsystem. Then* $m(v) \leqslant 3v+13$.

(iii) *Let* $v \equiv 7 \pmod{12}$, *and suppose that there exists a* KTS$(2v+13)$ *containing an* STS(v) *as a subsystem. Then* $m(v) \leqslant 3v+19$.

Proof. For (i), if an STS$(v=24t+15)$ can be embedded in a KTS$(2v+3=48t+33)$, embed the latter in an $S(2,4,72t+49=3v+4)$; this is possible by Theorem 6 since $2v+3 \equiv 9 \pmod{24}$. For (ii), if an STS$(v=24t+21)$ can be embedded in a KTS$(2v+9=48t+51)$, embed the latter in an $S(2,4,72t+76=3v+13)$ which is possible by Theorem 6 since $2v+9 \equiv 3 \pmod{24}$. For (iii), if an STS$(v=12t+7)$ can be embedded in a KTS$(2v+13=24t+27)$, embed the latter in an $S(2,4,36t+40=3v+19)$ which is possible by Theorem 6 since $2v+13 \equiv 3 \pmod{24}$. □

In the next section, we show some applications of Theorem 11 for small v. However, the lack of general results on embeddings of STSs into resolvable STSs (other than those given by Theorem 1 and Lemma 8) prevents us from applying Theorem 11 more widely.

3. Further necessary conditions

Suppose an STS(v) $(V,\mathscr{B})$ is embedded in an $S(2,4,w)$ $(W,\mathscr{C})$. Then the induced structure on the set $W \setminus V$ is a pairwise balanced design PBD$(w-v,\{3,4\},1)$ with Δ blocks of size three, and q blocks of size four. An easy calculation shows that $\Delta=(r'-r)v$ where $r=(v-1)/2$, $r'=(w-1)/3$, and $q=(\binom{w-v}{2}-3\Delta)/6$. In [4], necessary and sufficient conditions were obtained for a PBD$(v,\{3,4\},1)$ with Δ blocks of size 3 and q blocks of size 4 to exist (except for sufficiency in certain cases for small v, cf. [4]). We will make use of these necessary conditions in what follows.

Lemma 12. *Suppose* $v \equiv 9 \pmod{24}$, $v=24t+9$, *and suppose an* STS(v) *is embedded in an* $S(2,4,w)$. *Let* (i) $w=36t+13+12s$, *or* (ii) $w=36t+16+12s$. *Then* $s \geqslant 0$, *and*, *in case* (i),

$$12t^2+(4-24s)t+12s^2-14s \geqslant 0, \tag{1}$$

while in case (ii),

$$12t^2+(1-24s)t+12s^2-5s=1 \tag{2a}$$

or

$$12t^2+(1-24s)t+12s^2-5s \geqslant 6. \tag{2b}$$

Proof. By Lemma 5, we must have $s \geqslant 0$. In case (i), we have $w-v=12t+12s+4$, $r'-r=4s$, $\Delta=4s(24t+9)$, and $q=12t^2+(7-24s)t+12s^2-11s+1$. By [4], we must have

$q \geqslant (w-v)/4 = 3t+3s+1$ whence (1) follows. In case (ii), we have $w-v=12t+12s+7$, $r'-r=4s+1$, $\Delta=(4s+1)(24t+9)$, and $q=12t^2+(1-24s)t+12s^2-5s-1$. By [4], $q=0$ or $q \geqslant 5$ whence (2) follows. □

Lemma 13. *Suppose* $v \equiv 3 \pmod{24}$, $v=24t+3$, *and suppose an* STS(v) *is embedded in an* $S(2,4,w)$. *Let* (i) $w=36t+4+12s$, *or* (ii) $w=36t+13+12s$. *Then* $s \geqslant 0$, *and*, *in case* (i),

$$12t^2+(1-24s)t+12s^2-5s=0 \tag{3a}$$

or

$$12t^2+(1-24s)t+12s^2-5s \geqslant 5 \tag{3b}$$

while in case (ii),

$$12t^2-(20+24s)t+12s^2+10s \geqslant 0. \tag{4}$$

Proof. By Lemma 5, we must have $s \geqslant 0$. In case (i), we have $w-v=12t+12s+1$, $r'-r=4s$, $\Delta=4s(24t+3)$, and $q=12t^2+(1-24s)t+12s^2-5s$. By [4], $q=0$ or $q \geqslant 5$ whence (3) follows. In case (ii), $w-v=12t+12s+10$, $r'-r=4s+3$, $\Delta=(4s+3)(24t+3)$, and $q=12t^2-(17-24s)t+12s^2+13s+3$. By [4], $q \geqslant (w-v+2)/4=3t+3s+3$ whence (4) follows. □

The following lemmas give necessary conditions similar to those of the preceding two lemmas for remaining orders of Steiner triple systems. The proofs are also similar, and are therefore omitted.

Lemma 14. *Suppose* $v \equiv 7 \pmod{24}$, $v=24t+7$, *and suppose an* STS(v) *is embedded in an* $S(2,4,w)$. *Let* (i) $w=36t+13+12s$, *or* (ii) $w=36t+16+12s$. *Then* $s \geqslant 0$, *and*, *in case* (i),

$$12t^2-(4+24s)t+12s^2-6s \geqslant 3, \tag{5}$$

while in case (ii),

$$12t^2-(7+24s)t+12s^2+3s=1 \tag{6a}$$

or

$$12t^2-(7+24s)t+12s^2+3s \geqslant 6. \tag{6b}$$

Lemma 15. *Suppose* $v \equiv 13 \pmod{24}$, $v=24t+13$, *and suppose an* STS(v) *is embedded in an* $S(2,4,w)$. *Let* (i) $w=36t+25+12s$, *or* (ii) $w=36t+28+12s$. *Then* $s \geqslant 0$, *and*, *in case* (i),

$$12t^2-(4+24s)t+12s^2-6s \geqslant 5, \tag{7}$$

while in case (ii),

$$12t^2-(7+24s)t+12s^2+3s=2 \tag{8a}$$

or

$$12t^2 - (7 + 24s)t + 12s^2 + 3s \geqslant 7. \tag{8b}$$

Lemma 16. *Suppose* $v \equiv 15 \pmod{24}$, $v = 24t + 15$, *and suppose an* STS(v) *is embedded in an* $S(2,4,w)$. *Let* (i) $w = 36t + 25 + 12s$, *or* (ii) $w = 36t + 28 + 12s$. *Then* $s \geqslant 0$, *and, in case* (i),

$$12t^2 + (4 - 24s)t + 12s^2 - 14s \geqslant 3, \tag{9}$$

while in case (ii),

$$12t^2 + (1 - 24s)t + 12s^2 - 5s = 2 \tag{10a}$$

or

$$12t^2 + (1 - 24s)t + 12s^2 - 5s - 7 \geqslant 0. \tag{10b}$$

Lemma 17. *Suppose* $v \equiv 19 \pmod{24}$, $v=24t+19$, *and suppose an* STS(v) *is embedded in an* $S(2,4,w)$. *Let* (i) $w = 36t + 28 + 12s$, *or* (ii) $w = 36t + 37 + 12s$. *Then* $s \geqslant 0$, *and, in case* (i),

$$12t^2 + (17 - 24s)t + 12s^2 - 2s = 6 \tag{11a}$$

or

$$12t^2 + (17 - 24s)t + 12s^2 - 21s + 1 \geqslant 0, \tag{11b}$$

while in case (ii),

$$12t^2 - (4 + 24s)t + 12s^2 - 6s - 8 \geqslant 0. \tag{12}$$

Lemma 18. *Suppose* $v \equiv 21 \pmod{24}$, $v=24t+21$, *and suppose an* STS(v) *is embedded in an* $S(2,4,w)$. *Let* (i) $w = 36t + 37 + 12s$, *or* (ii) $w = 36t + 40 + 12s$. *Then* $s \geqslant 0$, *and, in case* (i),

$$12t^2 + (4 - 24s)t + 12s^2 - 14s \geqslant 5, \tag{13}$$

while in case (ii),

$$12t^2 + (1 - 24s)t + 12s^2 - 5s = 3 \tag{14a}$$

or

$$12t^2 + (1 - 24s)t + 12s^2 - 5s \geqslant 8. \tag{14b}$$

Lemma 19. *Suppose* $v \equiv 1 \pmod{24}$, $v = 24t + 1, t \geqslant 1$, *and suppose an* STS($v$) *is embedded in an* $S(2,4,w)$. *Let* (i) $w = 36t + 1 + 12s$, *or* (ii) $w = 36t + 4 + 12s$. *Then* $s \geqslant 0$, *and, in case* (i),

$$12t^2 - (4 + 24s)t + 12s^2 - 6s \geqslant 0, \tag{15}$$

while in case (ii),

$$12t^2 - (7 + 24s)t + 12s^2 + 3s = 0 \tag{16a}$$

or

$$12t^2 - (7 + 24s)t + 12s^2 + 3s - 5 \geqslant 0. \tag{16b}$$

The above lemmas will allow us to conclude that for no order of the form $v \equiv 3, 9 \pmod{24}$, $v \geqslant 9$ [and likely for many other orders] does the set $E(v)$ consist of an 'interval' of consecutive admissible orders w (i.e. admissible for the existence of an $S(2,4,w)$).

4. Embedding spectra for small orders

In this section we determine the embedding spectra $E(7)$ and $E(9)$, the latter with one possible exception.

Theorem 20. $E(7) = \{w : w \equiv 1, 4 \pmod{12}, w \geqslant 25\}$. *In other words, an* $S(2,4,w)$ *containing the* (*up to an isomorphism unique*) STS(7) *exists if and only if w is admissible and* $w \geqslant 25$.

Proof. If an STS(7) is embedded in an $S(2,4,w)$, we must have, by Corollary 6, $w \geqslant 13$. By Lemma 14, from (5) with $t = 0$, $s = 0$ we have $13 \notin E(7)$, and from (6) with $t = 0$, $s = 0$ we have $16 \notin E(7)$. (Of course, this also follows immediately from the well-known facts about the nonexistence of subplanes in PG(2,3) and AG(2,4), respectively.) On the other hand, in [9], Kramer et al. found $S(2,4,25)$ containing an STS(7) whence by Lemma 4, $q(v) \leqslant 76$ and thus $\{w : w \equiv 1, 4 \pmod{12}, w \geqslant 76\} \subseteq E(7)$. Thus it remains to be shown that $\{28, 37, 40, 49, 52, 61, 64, 73\} \subseteq E(7)$.

In [10], Krcadinac established that there are exactly 4466 $S(2,4,28)$s with nontrivial automorphism group. In the listing of these designs in http://student.math.hr/krcko/steiner.html, the following are the blocks of the design No.1:
$\{0,1,2,3\}, \{4,5,6,7\}, \{8,9,10,11\}, \{12,13,14,15\}, \{16,17,18,19\}, \{20,21,22,23\}, \{0,4,8,12\}, \{1,5,9,13\}, \{0,5,16,20\}, \{1,4,17,21\}, \{0,6,9,24\}, \{1,7,8,25\}, \{0,7,17,22\}, \{1,6,16,23\}, \{0,10,13,18\}, \{1,11,12,19\}, \{0,11,21,26\}, \{1,10,20,27\}, \{0,14,19,27\}, \{1,15,18,26\}, \{0,15,23,25\}, \{1,14,22,24\}, \{3,5,8,26\}, \{2,5,19,21\}, \{3,4,18,20\}, \{2,6,8,15\}, \{3,7,9,14\}, \{2,7,18,23\}, \{3,6,19,22\}, \{2,10,12,17\}, \{3,11,13,16\}, \{3,10,23,24\}, \{2,13,20,24\}, \{3,12,21,25\}, \{3,15,17,27\}, \{4,11,14,23\}, \{5,10,15,22\}, \{5,11,17,24\}, \{5,12,23,27\}, \{4,15,19,24\}, \{5,14,18,25\}, \{6,10,14,21\}, \{7,11,15,20\}, \{6,11,18,27\}, \{7,10,19,26\}, \{6,13,17,25\}, \{7,12,16,24\}, \{6,12,20,26\}, \{7,13,21,27\}, \{8,13,19,23\}, \{9,12,18,22\}, \{8,14,17,20\}, \{9,15,16,21\}, \{9,17,23,26\}, \{8,18,21,24\}, \{9,19,20,25\}, \{24,25,26,27\}, \{2,4,9,27\}, \{2,11,22,25\}, \{2,14,16,26\}, \{4,10,16,25\}, \{4,13,22,26\}, \{8,16,22,27\}$.

This design contains an STS(7) induced on the elements $\{2,4,16,22,25,26,27\}$ (as subsets of the last 7 blocks listed). Thus $28 \in E(7)$.

To show $37 \in E(7)$, consider the following $S(2,4,37)$ (with an automorphism of order 11): the elements are $Z_{11} \times \{1,2,3\} \cup \{\infty_1,\infty_2,\infty_3,\infty_4\}$, and the blocks are

$\{\infty_1,\infty_2,\infty_3,\infty_4\}$,
$\{0_1,0_2,0_3,\infty_1\} \bmod 11$,
$\{0_1,1_2,2_3,\infty_2\} \bmod 11$,
$\{0_1,2_2,5_3,\infty_3\} \bmod 11$,
$\{0_1,8_2,6_3,\infty_4\} \bmod 11$,
$\{0_1,1_1,5_1,10_2\} \bmod 11$,
$\{0_2,2_2,5_2,7_3\} \bmod 11$,
$\{8_1,0_3,1_3,5_3\} \bmod 11$,
$\{0_1,3_1,6_2,7_2\} \bmod 11$,
$\{0_2,4_2,8_3,10_3\} \bmod 11$,
$\{2_1,4_1,0_3,3_3\} \bmod 11$.

This design contains (e.g.) an STS(7) on the elements $\{0_1,1_1,2_2,10_2,3_3,4_3,5_3\}$.

To show $40 \in E(7)$, consider the cyclic $S(2,4,40)$ on Z_{40} with base blocks $\{0,1,4,13\},\{0,2,7,24\},\{0,6,14,25\},\{0,10,20,30\} \bmod 40$ (No.1 in the listing of [2]; the last of the base blocks generates the short orbit). This design contains an STS(7) on the elements $\{0,1,4,11,28,31,32\}$.

Similarly, to show that $49 \in E(7)$, consider the cyclic $S(2,4,49)$ on Z_{49} with base blocks $\{0,1,3,8\},\{0,4,20,30\},\{0,6,17,31\},\{0,9,21,36\} \bmod 49$ (No.5 in the listing of [2]). This design contains an STS(7) on the elements $\{0,1,3,26,42,44,45\}$. Furthermore, the cyclic $S(2,4,52)$ on Z_{52} with base blocks $\{0,1,3,31\},\{0,4,40,45\}$, $\{0,6,23,38\},\{0,8,33,42\},\{0,13,26,39\}, \bmod 52$ (design No.112 in the listing of [2]) contains an STS(7) on the set $\{0,1,3,20,28,37,49\}$; the cyclic $S(2,4,61)$ on Z_{61} with base blocks $\{0,1,3,8\}$, $\{0,4,13,31\},\{0,6,25,41\},\{0,10,24,39\}$, $\{0,11,23,44\} \bmod 61$ contains an STS(7) on the set $\{0,1,8,12,24,39,50\}$; the cyclic $S(2,4,64)$ on Z_{64} with base blocks $\{0,1,3,7\},\{0,5,17,39\},\{0,8,21,41\}$, $\{0,9,19,37\},\{0,11,26,40\},\{0,16,32,48\} \bmod 64$ contains an STS(7) on the set $\{0,3,7,18,33,45,56\}$. Finally, the cyclic $S(2,4,73)$ on Z_{73} with base blocks $\{0,1,3,7\},\{0,5,13,37\},\{0,9,26,55\},\{0,10,22,43\},\{0,11,25,45\}$, $\{0,15,31,50\} \bmod 73$ contains an STS(7) on the set $\{0,2,6,16,36,49,58\}$. All verifications are straightforward, and the proof of Theorem 20 is complete. □

Thus $m(7) = q(7) = 25$.

It is worth noting that Kramer et al. [9] and Spence [18], respectively, while determining that there are exactly 18 nonisomorphic designs $S(2,4,25)$, have also investigated, among other things, the existence of Fano planes (i.e. STS(7)) in these designs. They established that 5 of the 18 designs do not contain any STS(7) while 13 of the designs contain at least one STS(7).

We determined that of the 4466 nonisomorphic $S(2,4,28)$ with nontrivial automorphism group, exactly 1550 contain at least one STS(7). The number of nonisomorphic *cyclic* $S(2,4,v)$ for $v = 37,40,49,52,61,64,73$, and 76 is $2,10,224,206,18132,12048,1428546$, and 1113024, respectively. Of these, $0,4,31,8,743,379$, 40722, and 26863, respectively, contain at least one STS(7), while 2, 6, 193, 198, 17569, 11669, 1387824, and 1086161, respectively, do not contain any.

Remark. The total number of nonisomorphic cyclic $S(2,4,v)$ given above for $v=73$ and $v=76$ extends the enumeration results of [1,3] obtained earlier for cyclic $S(2,4,v)$ when $v \leqslant 64$.

Theorem 21. $E(9)=\{13,28\}\cup I\cup\{w : w\equiv 1,4\ (\mathrm{mod}\ 12), w\geqslant 40\}$ *where either* $I=\emptyset$ *or* $I=\{37\}$. *In other words, an* $S(2,4,w)$ *containing the* (*up to an isomorphism unique*) STS(9) *exists if and only if* $w=13$ *or* $w\geqslant 28$ *and admissible, except possibly when* $w=37$.

Proof. By Corollary 7, $m(9)\geqslant 13$. It is well known that the unique STS(9) (i.e. AG(2,3)) can be embedded in the unique $S(2,4,13)$ (i.e. PG(2,3)), thus $m(9)=13$. By Lemma 12, from (2) with $t=0,\ s=0$ we get $16\notin E(9)$ [again, it is a well-known fact that AG(2,4) does not contain AG(2,3)] and from (1) with $t=0,\ s=1$ we get $25\notin E(9)$. Consider the following $S(2,4,28)$ (design No.2611 in the listing of the 4466 $S(2,4,28)$ with nontrivial automorphism group by Krcadinac):
$\{1,17,18,22\},\{2,6,18,23\},\{3,6,17,24\},\{1,23,25,27\},\{2,24,25,26\},\{3,22,26,27\},\{0,1,4,7\},\{0,2,5,8\},\{0,3,6,9\},\{0,10,13,16\},\{0,11,14,17\},\{0,12,15,18\},\{0,19,22,25\},\{0,20,23,26\},\{0,21,24,27\},\{1,3,5,10\},\{1,2,6,11\},\{2,3,4,12\},\{1,8,9,13\},\{2,7,9,14\},\{3,7,8,15\},\{1,12,19,24\},\{2,10,20,22\},\{3,11,21,23\},\{1,15,16,26\},\{2,13,17,27\},\{3,14,18,25\},\{4,8,17,25\},\{5,9,18,26\},\{6,7,16,27\},\{4,9,23,24\},\{5,7,22,24\},\{6,8,22,23\},\{4,10,11,26\},\{5,11,12,27\},\{6,10,12,25\},\{7,12,13,23\},\{8,10,14,24\},\{9,11,15,22\},\{7,10,18,21\},\{8,11,16,19\},\{9,12,17,20\},\{7,11,20,25\},\{8,12,21,26\},\{9,10,19,27\},\{7,17,19,26\},\{8,18,20,27\},\{9,16,21,25\},\{10,15,17,23\},\{11,13,18,24\},\{12,14,16,22\},\{1,14,20,21\},\{2,15,19,21\},\{3,13,19,20\},\{4,6,18,19\},\{4,5,16,20\},\{5,6,17,21\},\{4,14,15,27\},\{5,13,15,25\},\{6,13,14,26\},\{4,13,21,22\},\{5,14,19,23\},\{6,15,20,24\}$.

This design contains an STS(9) induced on the set $\{4,5,6,13,14,15,19,20,21\}$ (as subsets of the last 12 blocks listed). Thus $28\in E(9)$. [In fact, we determined that exactly 128 of the 4466 nonisomorphic $S(2,4,28)$ with nontrivial automorphism group contain an STS(9)]. Invoking now the Rees–Stinson theorem completes the proof. □

Thus, in addition to $m(9)=13$, we have either $q(v)=28$ or $q(v)=40$. We conjecture that there exists an $S(2,4,37)$ containing an STS(9) (and thus $q(v)=28$), however, we have so far been unable to prove this. None of the cyclic $S(2,4,37)$s, nor any of the $S(2,4,37)$s with an automorphism of order 11, nor any of the many $S(2,4,37)$s with an automorphism of order 9 that we tested contains an STS(9) as a subsystem.

In what follows we investigate the embedding spectra $E(v)$ for other small values of v, namely $v\in\{13,15,19,21,25,27\}$. However, our results here are certainly far from best possible.

Theorem 22. (i) $37\leqslant m(13)\leqslant 40$.
(ii) $40\leqslant m(15)\leqslant 49$.

(iii) $52 \leqslant m(19) \leqslant 85$.
(iv) $61 \leqslant m(21) \leqslant 76$.
(v) $40 \leqslant m(25) \leqslant 76$; *furthermore*, $49, 52 \notin E(25)$.
(vi) $m(27) = 40$.

Proof. (i) By Corollary 7, $m(13) \geqslant 25$. By Lemma 15, we get from (7) with $t=0$, $s=0$ that $25 \notin E(13)$, and from (8) with $t=0$, $s=0$ that $28 \notin E(13)$. Thus $m(13) \geqslant 37$. By Theorem 9(ii), $m(13) \leqslant 40$.

(ii) By Corollary 7, $m(15) \geqslant 25$. By Lemma 16, we get from (9) with $t=0$, and with $s=0$ and 1, respectively, that $25 \notin E(15)$, and $37 \notin E(15)$, respectively; from (10) we get with $t=0$, $s=0$ that $28 \notin E(15)$. Thus $m(15) \geqslant 40$. By Theorem 11(i), in order to show that $m(15) \leqslant 49$, it suffices to show that there exists a KTS(33) with a sub-STS(15). The following is such a KTS(33) on the set of elements $V = Z_{15} \times \{1,2\} \cup \{\infty_1, \infty_2, \infty_3\}$:

Fifteen parallel classes are obtained by developing modulo 15 the base parallel class $\{0_1, 7_1, 7_2\}, \{1_1, 4_1, 13_2\}, \{2_1, 8_1, 3_2\}, \{3_1, 5_1, 11_2\}, \{10_1, 14_{,1}, 12_2\}, \{12_1, 13_1, 2_2\}, \{0_2, 1_2, 4_2\}, \{6_2, 8_2, 14_2\}, \{\infty_1, 6_1, 9_2\}, \{\infty_2, 9_1, 5_2\}, \{\infty_3, 11_1, 10_2\}$. The 16th parallel class is obtained by developing, modulo 15, $\{0_i, 5_i, 10_i\}$, $i = 1, 2$, together with $\{\infty_1, \infty_2, \infty_3\}$. The sub-STS(15) on the set $Z_{15} \times \{2\}$ is generated by the base blocks $\{0_2, 1_2, 4_2\}, \{6_2, 8_2, 14_2\}, \{0_2, 5_2, 10_2\}$.

The lower bounds in (iii)–(v) are obtained by applying Corollary 7 and inequalities (11)–(16). The inequalities (15) and (16) also imply $49, 52 \notin E(25)$. The upper bounds in (iii) and (v) are obtained from Theorem 9, and we have (vi) by Theorem 6. Finally, to show the upper bound in (iv), namely $m(21) \leqslant 76$, by Theorem 11(ii) it suffices to show that there exists a KTS(51) with a sub-STS(21). The following is such a KTS(51) on the set $Z_{21} \times \{1,2\} \cup \Omega$ where $\Omega = \{\infty_i : i = 1, 2, \ldots, 9\}$:

Twenty-one parallel classes are obtained by developing modulo 21 the base parallel class $\{0_1, 5_1, 9_2\}, \{3_1, 9_1, 11_2\}, \{6_1, 18_1, 7_2\}, \{7_1, 11_1, 18_2\}, \{16_1, 19_1, 1_2\}, \{1_1, 2_1, 14_2\}, \{2_2, 5_2, 17_2\}, \{0_2, 16_2, 20_2\}, \{\infty_1, 4_1, 4_2\}, \{\infty_2, 8_1, 13_2\}, \{\infty_3, 10_1, 3_2\}, \{\infty_4, 12_1, 6_2\}, \{\infty_5, 13_1, 8_2\}, \{\infty_6, 14_1, 10_2\}, \{\infty_7, 15_1, 12_2\}, \{\infty_8, 17_1, 15_2\}$, $\{\infty_9, 20_1, 19_2\}$.

Three further parallel classes are obtained as follows. If R_0, R_1, R_2, R_3 are the four parallel classes of an STS(9) on the set Ω, form, for $j = 0, 1, 2$, the parallel class $P_j = \{\{3x + j_i, 3x + j + 2_i, 3x + j + 10_i\} : x = 0, 1, \ldots, 6; i = 1, 2\} \cup R_j$ (modulo 21, of course).

The last parallel class is obtained as $\{\{0_i, 7_i, 14_i\} : i = 1, 2\} \cup R_3$, again modulo 21. This completes the proof. □

Corollary 23. (i) $q(13) \leqslant 121$,
(ii) $q(15) \leqslant 148$; *furthermore*, $\{76, 85, 112, 121\} \subset E(15)$,
(iii) $q(19) \leqslant 256$; *furthermore*, $\{184, 193, 220, 229\} \subset E(19)$,
(iv) $q(21) \leqslant 229$, *furthermore*, $\{112, 121, 148, 157, 184, 193, 220\} \subset E(21)$,
(v) $q(25) \leqslant 229$; *furthermore*, $112 \in E(25)$, *and*
(vi) $73 \leqslant q(27) \leqslant 121$.

Proof. The upper bound inequalities follow directly from Theorem 22, Lemma 4 and Corollary 10. Inequalities (3) and (4) of Lemma 13 show that $49,52,61,64 \notin E(27)$, and thus $q(27) \geqslant 73$. The rest follows by combining Theorems 1, 2, and 6, and Lemma 8. □

5. Conclusion

Even though for every fixed order v there is only a finite number of orders w for which one has to decide whether there exists an embedding of some STS(v) in an $S(2,4,w)$, there seems to be no easy way to determine the sets $E(v)$, even for relatively small orders v. It appears that the customary design-theoretic arsenal that has expanded so dramatically during the recent decades, still needs to be further developed to be able to handle this (what we believe is a) new kind of design embedding.

As already mentioned in the introduction, one property absent here is the replacement property. In fact, it is easy to see that for given orders v and w, there may exist both, an STS(v) embeddable in an $S(2,4,w)$, and an STS(v) not embeddable in any $S(2,4,w)$. Currently, smallest such examples are provided by resolvable and non-resolvable STS(27): the former can be embedded in an $S(2,4,40)$ while the latter cannot.

There are many questions that relate embeddings of STSs into $S(2,4,w)$s to outstanding colouring problems. For example, the existence of an embedding of an STS(39) into an $S(2,4,61)$ could shed some light on the existence of a 3-colouring of type B (cf. [13]) of the latter, while M.J. de Resmini's question about the existence of the 'century design' is equivalent to the question of the existence of an embedding of an STS(45) [or of an STS(55)] into an $S(2,4,100)$.

Let us conclude with a result which at first glance appears quite strong but is in fact just a stronger version of Theorem 3 and follows equally easily from Ganter's Theorem.

Theorem 24. *For any* STS(v) $(V,\mathscr{B})$ *there exists an infinite sequence* $((V_i,\mathscr{B}_i) : i = 0,1,\ldots)$ *where* $(V_0,\mathscr{B}_0)=(V,\mathscr{B})$, $(V_i,\mathscr{B}_i)$ *is a Steiner system* $S(2,i+3,w_i)$, *and for each* $i=0,1,\ldots$, *the Steiner system* $(V_i,\mathscr{B}_i)$ *can be embedded in a Steiner system* $(V_{i+1},\mathscr{B}_{i+1})$.

However, as things stand now, each successive embedding is exponential in terms of the preceding order w_i. A first step towards improving this unfortunate state of affairs would be to come up with a polynomial (or at least a subexponential) embedding of a partial $S(2,4,v)$ into an $S(2,4,w)$.

Acknowledgements

Thanks to Curt Lindner and Peter Horak for several valuable comments, and to Giovanni LoFaro for pointing out an omission in the necessary conditions of Section

3. Most of this research was done while the first author was a postdoctoral fellow in the Department of Mathematics and Statistics, McMaster University. Research of the second author was supported by NSERC of Canada Grant No. OGP007268.

References

[1] M.J. Colbourn, Cyclic block designs: computational aspects of their construction and analysis, Ph.D. Thesis, University of Toronto, 1980.

[2] M.J. Colbourn, R.A. Mathon, On cyclic Steiner 2-designs, Ann. Discrete Math. 7 (1980) 215–253.

[3] C.J. Colbourn, A. Rosa, Triple Systems, Oxford University Press, Oxford, 1999.

[4] C.J. Colbourn, A. Rosa, D.R. Stinson, Pairwise balanced designs with block sizes three and four, Canad. J. Math. 43 (1991) 673–704.

[5] J. Doyen, R.M. Wilson, Embeddings of Steiner triple systems, Discrete Math. 5 (1973) 229–239.

[6] B. Ganter, Partial pairwise balanced designs, Teorie Combinatorie II, Colloq. Internaz. Roma 1973, Atti dei Convegni Lincei 17, Accad. Naz. dei Lincei, Roma, 1976, pp. 377–380.

[7] A. Hartman, Quadruple systems containing $AG(3,2)$, Discrete Math. 39 (1982) 293–299.

[8] J.W.P. Hirschfeld, Projective Geometries over Finite Fields, Oxford University Press, Oxford, 1985.

[9] E.S. Kramer, S.S. Magliveras, R. Mathon, The Steiner systems $S(2,4,25)$ with nontrivial automorphism group, Discrete Math. 77 (1989) 137–157.

[10] V. Krcadinac, The $2-(28,4,1)$ designs with nontrivial automorphism group, Glasnik Mat., to appear.

[11] M. Limbos, Projective embeddings of small “Steiner triple systems”, Ann. Discrete Math. 7 (1980) 151–173.

[12] R.A. Mathon, K.T. Phelps, A. Rosa, A class of Steiner triple systems of order 21 and associated Kirkman systems, Math. Comput. 37 (1981) 209–222.

[13] S. Milici, A. Rosa, V. Voloshin, Colouring Steiner triple systems with prescribed block colour patterns, Discrete Math. 240 (2001) 145–160.

[14] R.C. Mullin, D.R. Stinson, S.A. Vanstone, Kirkman triple systems containing maximum subdesigns, Utilitas Math. 21C (1982) 283–300.

[15] R. Rees, D.R. Stinson, On the existence of Kirkman triple systems containing Kirkman subsystems, Ars Combin. 26 (1988) 3–16.

[16] R. Rees, D.R. Stinson, On the existence of incomplete block designs of block size four having one hole, Utilitas Math. 35 (1989) 119–152.

[17] M.J. de Resmini, On sets of type (m,n) in BIBD's with $\lambda \geqslant 2$, Ann. Discrete Math. 14 (1982) 183–206.

[18] E. Spence, Complete classification of Steiner systems $S(2,4,25)$, J. Combin. Designs 4 (1996) 295–300.