On Kirkman packing designs KPD ($\{3, 4\}$, $v$)

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Abstract

A Kirkman packing design KPD ($\{3, 4^*\}$, $v$) is a resolvable packing of a $v$-set by the maximum possible number of parallel classes, each containing two blocks of size 4 and all other blocks of size three. Such designs can be used to construct certain threshold schemes. In this paper, direct and recursive constructions are discussed for such designs. The existence of a KPD ($\{3, 4^*\}$, $v$) is established for every $v \equiv 2 \pmod{3}$ with $v \geq 32$.

Key words: Kirkman packing design, threshold scheme, Kirkman frame

1 Introduction

A packing of $X$ is a collection of subsets of $X$ (called blocks) such that any pair of distinct points from $X$ occur together in at most one block in the collection. A packing is called resolvable if its block set admits a partition into parallel classes, each parallel class being a partition of the point set $X$.

A Kirkman packing design, denoted KPD ($K, v$), is a resolvable packing of a $v$-set by the maximum possible number $m(v)$ of parallel classes, each containing the same number of blocks of each size in $K$. If there is only one block of size $s$ in every parallel class of KPD, and all others have block size $w$, we denoted it by KPD ($\{w, s^*\}$, $v$). In [4, 6], such a design is also called Kirkman school project design when $K = \{3, s\}$, $s \in \{2, 4\}$. When $K = \{3\}$, such a design is called Kirkman Triple System KTS($v$) if $v \equiv 3 \pmod{6}$ or Nearly Kirkman Triple System NKTS($v$) if $v \equiv 0 \pmod{6}$.

It is shown in [3, Lemma 2.1] that a KPD ($\{w, s^*\}$, $v$) can be used to construct a (2,$w$)-threshold scheme when $s \geq w$. In this scheme, the number of keys is the number of parallel classes in the KPD. In fact, any KPD ($K, v$) can be used to construct a (2,$w$)-threshold
scheme if $k \geq w$ for any $k \in K$. The proof is similar to the proof of Lemma 2.1 in [3]. So, for given $w$ and $v$, we need to minimize the number of pairs of points contained in each parallel class, if we want to construct a threshold scheme having maximum number of keys. Let $u \equiv v \pmod{w}$. It's not difficult to show that if $K = \{w, w + 1\}$ and any parallel class contains $u$ blocks of size $w + 1$ and $(v - u(w + 1))/w$ blocks of size $w$, then the number of keys is maximum.

In this paper, we consider $w = 3$. The known results concerning KPD $(\{3, s\}, v)$ for $s = 3, 4, v \equiv 0, 1 \pmod{3}$, are as following.

**Theorem 1.1** [10] There exists an KTS $(v)$ containing $\frac{v-1}{2}$ parallel classes if and only if $v \equiv 3 \pmod{6}$.

**Theorem 1.2** [8, 1, 2, 12] There exists an NKTS $(v)$ containing $\frac{v-2}{2}$ parallel classes if and only if $v \equiv 0 \pmod{6}$ and $v \geq 18$.

**Theorem 1.3** [4, 9, 6, 3] There is a KPD $(\{3, 4^*\}, v)$ containing $\left\lfloor \frac{v-3}{2} \right\rfloor$ parallel classes for every $v \equiv 1 \pmod{3}$ with $v \geq 25$.

For $v \equiv 2 \pmod{3}$, as discussed above, we need to construct the KPDs which contain 2 blocks of size 4 and $(v - 8)/3$ blocks of size three in each parallel class. For convenience, we denote the KPD by KPD $(\{3, 4^*\}, v)$. We shall establish the existence of a KPD $(\{3, 4^*\}, v)$ for every $v \equiv 2 \pmod{3}$ with $v \geq 32$.

**Theorem 1.4** There exists a KPD $(\{3, 4^*\}, v)$ containing $\left\lfloor \frac{v-5}{2} \right\rfloor$ parallel classes for every $v \equiv 2 \pmod{3}$ with $v \geq 32$.

## 2 Frames and IKPDs

In this section we will give some small designs for our recursive constructions. We generalize the idea of [6, 9, 3] and construct incomplete Kirkman packing designs IKPD $(\{3, 4^*\}, v, w)$. We shall also use frames in both direct and recursive constructions.

A group-divisible design (GDD) is a triple $(X, G, B)$ which satisfies the following properties: (i) $X$ is a finite set of points, (ii) $G$ is a partition of $X$ into subsets called groups, (iii) $B$ is a set of subsets of $X$ (called blocks), such that a group and a block contain at most one common point, and every pair of points from distinct groups occur in exactly one block.
The type of a GDD is the multiset \(|G| : G \in \mathcal{G}\). We denote the type by \(1^{u_1}2^{u_2}\cdots\), where there are precisely \(u_i\) occurrences of \(i\), \(i \geq 1\). The set of block sizes is denoted by \(K\). A transversal design \(TD(k, n)\) is a \(k\)-GDD of type \(n^k\). It is idempotent if it contains a parallel class of blocks.

A GDD\((X, \mathcal{G}, B)\) is called frame resolvable if its block set \(B\) can be partitioned into frame parallel classes, each class being a partition of \(X - G_j\) for some \(G_j \in \mathcal{G}\). A Kirkman frame is a frame resolvable GDD in which all the blocks have size three. It is well known that to each \(G_j\) there are exactly \(|G_j|/2\) frame parallel classes of triples that partition \(X - G_j\). The groups in a Kirkman frame are often referred to as holes. From Stinson [13], we have the following result.

**Theorem 2.1** There exists a Kirkman frame of type \(g^u\) if and only if \(u \geq 4\), \(g\) is even and \(g(u - 1) \equiv 0 \pmod{3}\).

In order to use the ‘Filling in Holes’ construction, we need the notion of an incomplete Kirkman packing design (IKPD). Let \(v \equiv h \equiv 2\) or \(5 \pmod{6}\). For \(h \geq 5\), an IKPD \((\{3, 4^*\}, v, h)\) is defined to be a triple \((V, H, B)\) which satisfies the following properties:

1. \(V\) is a \(v\)-set of points, \(H\) is a \(h\)-subset of \(V\) (called a hole) and \(B\) is a collection of subsets of \(V\) (called blocks), each block having size 3 or 4;
2. \(|H \cap B| \leq 1\) for all \(B \in B\);
3. any two points of \(V\) appear either in \(H\) or in at most one block of \(B\);
4. \(B\) admits a partition into \((v - h)/2\) parallel classes, each consisting of two blocks of size 4 and \((v - 8)/3\) triples on \(V\), and \([h - 5]/2\) auxiliary parallel classes, each consists of \((v - h)/3\) triples on \(V \setminus H\).

In what follows, we take the point set \(V = Z_t \times \{1, 2\} \cup H, t = (v - h)/2\). The \((v - h)/2\) parallel classes will be generated mod \(t\) from an initial parallel class \(P\). The auxiliary parallel classes will be described separately in each case.

**Lemma 2.2** There is an IKPD \((\{3, 4^*\}, 32, 8)\).

**Proof:** Take the point set \(V = Z_{12} \times \{1, 2\} \cup H, H = \{\infty_1, \cdots, \infty_6\} \cup \{a_0, a_1\}\). Let \(B = \{0, 4, 8\}, B \times \{1\}\) and \(B \times \{2\}\) together generate the required auxiliary parallel class by +3 modulo 12. \(P\) contains the following blocks, where the subscripts on \(a\) are evaluated mod 2.

\[
\begin{align*}
1_12_14_13_2 &\quad 9_12_22_24_2 &\quad 3_18_1a_0 &\quad 5_20_2a_1 &\quad 0_18_2\infty_1 \\
5_11_2\infty_2 &\quad 6_16_2\infty_3 &\quad 7_110_2\infty_4 &\quad 10_17_2\infty_5 &\quad 11_19_2\infty_6
\end{align*}
\]
Lemma 2.3 There is an IKPD ($\{3, 4^{\ast\ast}\}, v, 8$) for $v \in \{38, 44, 50, 56\}$.

Proof: Take the point set $V = Z_t \times \{1, 2\} \cup H, t = (v - 8)/2, H = \{\infty_1, \ldots, \infty_8\}$. Let $B = \{0, t/3, 2t/3\}, B \times \{1\}$ and $B \times \{2\}$ together generate the required auxiliary parallel class by $+3$ modulo $t$, except where $v = 44$, in which case the auxiliary parallel class is given by $i_1(i + 6)_1(i + 12)_1, i_2(i + 6)_2(i + 12)_2 : 0 \leq i \leq 5$.

For each $v$, $P$ contains the following blocks.

$v = 38 : \begin{align*}
121 & 5132 & 612 & 252 & 319 & 111 & 426 & 212 & 4110 & 2 \infty_1 & 7111 \infty_2 \\
810 & 2 \infty_3 & 10113 & 2 \infty_4 & 12192 \infty_5 & 13172 \infty_6 & 141142 \infty_7 & 0182 \infty_8 
\end{align*}$

$v = 44 : \begin{align*}
121 & 5132 & 612 & 252 & 3181 & 101 & 4292 & 112 & 71151 & 122 & 6216 \infty_1 \\
41142 & \infty_1 & 91152 & \infty_2 & 11102 & \infty_3 & 131132 & \infty_4 & 141172 & \infty_5 & 16172 & \infty_6 \\
171102 & \infty_7 & 0182 & \infty_8 
\end{align*}$

$v = 50 : \begin{align*}
121 & 1232 & 252 & 7191 & 3161 & 111 & 4181 & 141 & 7215 & 2162 & 4282 & 142 \\
12101 & 92 & 132182 & 101 & 51172 & \infty_1 & 131202 & \infty_2 & 151192 & \infty_3 & 16102 & \infty_4 \\
17162 & \infty_5 & 18112 & \infty_6 & 19112 & \infty_7 & 201102 & \infty_8 
\end{align*}$

$v = 56 : \begin{align*}
121 & 1232 & 252 & 7191 & 3161 & 101 & 4113 & 181 & 10216 & 232 & 4282 & 92 \\
51161 & 192 & 11117 & 212 & 72172 & 201 & 15202 & 81 & 12182 & \infty_1 & 141222 & \infty_2 \\
151 & 202 & \infty_3 & 191132 & \infty_4 & 21162 & \infty_5 & 221112 & \infty_6 & 231142 & \infty_7 & 01122 & \infty_8 
\end{align*}$

Lemma 2.4 There is an IKPD ($\{3, 4^{\ast\ast}\}, 146, 32$).

Proof: Take the point set $(Z_{57} \times \{1, 2\}) \cup H, H = (\{a\} \times Z_3) \cup \{\infty_1, \ldots, \infty_9\}$. For each $B \in \{0, 4, 11, 0, 5, 13, 0, 10, 26, 0, 14, 34\}, B \times \{1\}$ and $B \times \{2\}$ together generate three auxiliary parallel classes by $+1$ modulo $57$ since each block contains three elements different modulo 3. This gives the twelve auxiliary parallel classes. The thirteenth auxiliary parallel class is generated $+3$ modulo $57$ by $\{01191381, 02192382\}$. $P$ contains the following blocks, where the subscripts on $a$ are evaluated mod 3.
Lemma 2.5 There is an IKPD (\{3, 4^{**}\}, 89, 23).

Proof: Take the point set \((Z_{33} \times \{1, 2\}) \cup H, H = (\{a\} \times Z_3) \cup \{\infty_1, \cdots, \infty_{20}\}\). For each \(B \in \{0 1 5, 0 2 10\}, B \times \{1\}\) and \(B \times \{2\}\) together generate three auxiliary parallel classes by +1 modulo 33 since each block contains three elements different modulo 3. Let \(B_1 = \{1 8 11\}\) and \(B_2 = \{3 1 2\}\). It’s easy to see that \(B_1\) and \(B_2\) also generate three auxiliary parallel classes by +1 modulo 33. Thus we get the required nine auxiliary parallel classes. \(P\) contains the following blocks, where the subscripts on \(a\) are evaluated mod 3.

\[
\begin{align*}
112411 & \quad 31425272 & 53192342 & 24130521 & 42121112 & 151361451 \\
1420472 & \quad 32441272 & 10128532 & 18132392 & 14131452 & 51381512 \\
11501382 & \quad 1316242 & 612112 & 8252372 & 7139100 & 92312 & 1 \\
19152a2 & \quad 81132 & 91162 & 1211023 & 161262 & 171292 & 5 \\
201232 & \quad 221412 & 231432 & 251542 & 261562 & 271522 & 11 \\
2815202 & \quad 29102 & 331492 & 341422 & 351172 & 371402 & 17 \\
4015202 & \quad 41182 & 431212 & 461362 & 471222 & 481402 & 23 \\
491322 & \quad 511302 & 541482 & 551352 & 561322 & 014422 & 29 \\
\end{align*}
\]

In the remaining part of this section, we will use a small variant of the above constructions such that the \(r\)th parallel class \(P_r\) will consist of two parts \(Q_r\) and \(F_r\), \(0 \leq r \leq t - 1\). The main part \(Q_r\) will be generated from \(Q_0\) modulo \(t\). We assume that \(t = kn, k \equiv 1 (mod 3)\) and \(k \geq 4\) so that the \(i\)th Kirkman frame of type \(2^k\) is based on the set \(\{i, n+i, \cdots, (k-1)n+i\} \times \{1, 2\}\). Suppose \(r = jn + i, 0 \leq j \leq k-1\) and \(0 \leq i \leq n - 1\). Take \(F_r\) to be the frame parallel class with the hole \(\{r\} \times \{1, 2\}\) in the \(i\)th Kirkman frame. In the following constructions, we need to list the parameters \(v\), \(h\) and \(k\), and the blocks of \(Q_0\). Note that \(Q_0\) contains all points of \(V\) except \(2k - 2\) points in \(\{n, 2n, \cdots, (k-1)n\} \times \{1, 2\}\).

Lemma 2.6 There is an IKPD (\{3, 4^{**}\}, \v, 5) for \(\v \in \{53, 101\}\).

Proof: Take the point set \((Z_i \times \{1, 2\}) \cup H, t = (v - 5)/2, H = \{\infty_1, \cdots, \infty_5\}\). For each \(v\), \(P\) contains the following blocks.
There is an IKPD evaluated mod 4. Let $v \in \{0, 1, 3, 4\}$.

**Proof:** An IKPD ($\{3, 4^{**}\}$) has no auxiliary parallel classes, so it's easy to check that those KPDs ($\{3, 4^{**}\}$) for $v = 23, 35, 41, 47, 53, 59$ constructed in Appendix B are evaluated mod 4.

For later use, we combine these IKPDs in the following.

**Lemma 2.8** (i) There is an IKPD ($\{3, 4^{**}\}, v, 5$) for $v = 23, 35, 41, 47, 53, 59$.

(ii) There is an IKPD ($\{3, 4^{**}\}, v, 8$) for $v = 32, 38, 44, 50, 56$. 

**Proof:** An IKPD ($\{3, 4^{**}\}, v, h$) for $h = 5$ has no auxiliary parallel classes, so it's easy to check that those KPDs ($\{3, 4^{**}\}, v$) for $v = 23, 35, 41, 47, 59$ constructed in Appendix B are...
indeed IKPDs. An IKPD \(\{3,4^{**}\}, 53,5\) comes from Lemma 2.6. (ii) comes from Lemma 2.2 and Lemma 2.3.

Lemma 2.9 There is a KPD \(\{3,4^{**}\}, v\) for \(v = 32, 38, 44, 50, 56\).

Proof: Note that a KPD \(\{3,4^{**}\}, v\) has \((v−6)/2\) parallel classes and an IKPD \(\{3,4^{**}\}, v, 8\) has \((v−8)/2\) parallel classes and one auxiliary parallel class. Since no pairs of points in the “hole” are used in the IKPD, we can partition the points in the “hole” into two blocks and add them to the auxiliary parallel class to form a new parallel class, so we get \((v−8)/2 + 1 = (v−6)/2\) parallel classes which can form a KPD \(\{3,4^{**}\}, v\). Thus this lemma comes from Lemma 2.8 (ii).

For convenience, we also combine these KPDs in the following Lemma.

Lemma 2.10 (i) There is a KPD \(\{3,4^{**}\}, v\) for \(v = 62, 68, 74, 80, 86, 92, 98, 110, 116, 122, 146\). (ii) There is a KPD \(\{3,4^{**}\}, v\) for \(v = 23, 35, 41, 47, 53, 59, 65, 71, 83, 89, 101, 107, 119, 137\).

Proof: From Lemma 2.4, we have an IKPD \(\{3,4^{**}\}, 146,32\), filling in the “hole” with a KPD \(\{3,4^{**}\}, 32\), then we obtain a KPD \(\{3,4^{**}\}, 146\). Similarly we can get a KPD \(\{3,4^{**}\}, 89\) where an IKPD \(\{3,4^{**}\}, 89, 23\) comes from Lemma 2.5 and a KPD \(\{3,4^{**}\}, 23\) listed in Appendix B. With an IKPD \(\{3,4^{**}\}, 137, 41\) from Lemma 2.7 and a KPD \(\{3,4^{**}\}, 41\) from Appendix B, we solve the order 137. The other orders in (i) come from Appendix A. A KPD \(\{3,4^{**}\}, v\) for \(v = 53, 101\) comes from Lemma 2.6 since an IKPD \(\{3,4^{**}\}, v, 5\) is indeed a KPD \(\{3,4^{**}\}, v\). The other orders in (ii) come from Appendix B.

3 Main results

In this section, we show our main results. We shall use the following ‘Filling in Holes’ construction.

Lemma 3.1 If there exist a Kirkman frame of type \(g_1 \cdots g_u\), IKPD \(\{3,4^{**}\}, g_i + h, h\)s for \(1 \leq i < u\), and a KPD \(\{3,4^{**}\}, g_u + h\), then a KPD \(\{3,4^{**}\}, h + \sum_{i=1}^{u} g_i\) exists.

Proof: For \(1 \leq i < u\), there are \(g_i/2\) frame parallel classes missing the group of size \(g_i\), and the same number of parallel classes in the IKPD \(\{3,4^{**}\}, g_i + h, h\) which contains two
blocks of size four; match them up arbitrarily, placing the \( g_i \) points of the IKPD on the \( i \)th group of the frame and the \( h \) points in its hole on \( h \) new points.

Next, each IKPD contains \([ (h - 5)/2 \] auxiliary parallel classes of triples. From unions of these with \([ (h - 5)/2 ]\) parallel classes of the KPD \((\{3, 4^{**}\}, g_u + h)\), to form \([ (h - 5)/2 ]\) additional parallel classes. There remain \( g_u/2 \) parallel classes of the KPD \((\{3, 4^{**}\}, g_u + h)\), which can be matched arbitrarily with the \( g_u/2 \) frame parallel classes of the \( u \)th group to complete the construction.

In order to use the ‘Filling in Holes’ construction illustrated above, we will require Kirkman frames in which the holes are not necessarily all of the same size. To get these, we use the following ‘Weighting’ construction (see, e.g. Stinson [13]).

**Lemma 3.2** Suppose that there is a \( K\)-GDD of type \( g_1^{t_1}g_2^{t_2}\) \( \cdots \) \( g_m^{t_m} \) and that for each \( k \in K \) there is a Kirkman frame of type \( h^k \). Then there is a Kirkman frame of type \((hg_1)^{t_1}(hg_2)^{t_2}\) \( \cdots \) \((hg_m)^{t_m}\).

**Lemma 3.3** For each \( v, v \equiv 0 \pmod{6} \), \( v \geq 234 \), there is a Kirkman frame of type \( 42^a36^b30^c \), where \( v = 42a + 36b + 30c \), \( a \geq 4 \), \( b, c \geq 0 \) or \( a = 0 \), \( b \geq 4 \), \( c \geq 0 \)

**Proof:** Let \((n_0, n_1, \cdots)\) be the infinite sequence of integers defined as follows. The initial sequence \((n_0, \cdots, n_9)\) is \((7, 9, 11, 13, 17, 23, 27, 37, 47, 63)\). Let \( n_{9+j} = 63 + j \) for \( j \geq 1 \). From [5, p.126], an idempotent TD \((7, n_i)\) exists for each \( n_i \), \( i \geq 0 \). Let \( t = v/6 \). Since \( v \geq 234 \), we have \( t \geq 39 \). There exists an integer \( n_i \) from the sequence so that \( 5n_i + 4 \leq t \leq 7n_i \). This can always be done because \( 7n_i \geq 5n_i + 4 \) for all \( i \geq 0 \). Let \( t = 7a + 6b + 5c \), where \( a = n_i - b - c \), \( 0 \leq b \leq n_i - 4 \) when \( a > 0 \) and \( 4 \leq b \leq n_i \) when \( a = 0 \). Form the idempotent TD \((7, n_i)\) with groups \( G_1, \cdots, G_7 \) and blocks \( B_1, \cdots, B_{n_i} \) in one parallel class. Delete \( n_i - a \) points in \( G_7 \) that lie in \( B_{a+1}, \cdots, B_{n_i} \). Furthermore, delete \( c \) points in \( G_6 \) that lie in \( B_{n_i-c+1}, \cdots, B_{n_i} \). Taking the truncated blocks \( B_1, \cdots, B_{n_i} \) as groups, we have formed a GDD of type \( 7^a6^b5^c \), having all blocks of size at least four. We may apply Lemma 3.2 with weight \( h = 6 \) to obtain a Kirkman frame of type \( 42^a36^b30^c \). 

To get more Kirkman frames we also need the following results on 4-GDD of type \( g^4m^1 \). From Rees [11] and [7] we have:

**Theorem 3.4** There exists a 4-GDD of type \( g^4m^1 \) with \( m > 0 \) if and only if \( g \equiv m \equiv 0 \pmod{3} \) and \( 0 < m \leq 3g/2 \).
Lemma 3.5 There exists a Kirkman frame of type $(2g)^4(2m)^1$ with $m > 0$ if and only if \( g \equiv m \equiv 0 \pmod{3} \) and $0 < m \leq 3g/2$.

Proof: From Theorem 3.4 there exist a 4-GDD of type $g^4m^1$. Applying Lemma 3.2 with $h = 2$ and $k = 4$ we get the required Kirkman frame, as a Kirkman frame of type $2^4$ comes from Theorem 2.1.

Lemma 3.6 There is a $KPD (\{3,4^{**}\}, v)$ for $v \equiv 2 \pmod{3}, v \geq 239$.

Proof: By Lemma 3.3 there exist Kirkman frame of type $42^a36^bc^e$ on $v'$ points, $v' \equiv 0 \pmod{6}$ and $v' \geq 234$, where $a \geq 4, b, c \geq 0$ or $a = 0, b \geq 4, c \geq 0$. For $v \equiv 2 \pmod{6}$, apply Lemma 3.1 with $h = 8$. Use IKPD $(\{3,4^{**}\}, 50, 8)$, IKPD $(\{3,4^{**}\}, 44, 8)$ and IKPD $(\{3,4^{**}\}, 38, 8)$ from Lemma 2.8 to fill all but one hole of size $6u \in \{30, 36, 42\}$, and fill the final hole with a KPD $(\{3,4^{**}\}, 6u + 8)$ from Lemma 2.9. Then we obtain a KPD $(\{3,4^{**}\}, v)$, where $v = v' + 8$. For $v \equiv 5 \pmod{6}$, again apply Lemma 3.1 with $h = 5$. Use IKPD $(\{3,4^{**}\}, 47, 5)$, IKPD $(\{3,4^{**}\}, 41, 5)$ and IKPD $(\{3,4^{**}\}, 35, 5)$ from Lemma 2.8 to fill all but one hole of size $6u$. Then, fill the final hole with a KPD $(\{3,4^{**}\}, 6u + 5)$ from Lemma 2.10 to obtain a KPD $(\{3,4^{**}\}, v)$, where $v = v' + 5$.

Now we are in a position to prove Theorem 1.4.

Proof of Theorem 1.4: By Lemma 3.6 it remains to consider the orders $v \leq 236$. For $v$ odd and $v = 227, 233$, with frames of types $48^436^1$ and $48^430^1$ from Lemma 3.5, we apply Lemma 3.1 with $h = 5$ to fill in “holes” using IKPD $(\{3,4^{**}\}, 53, 5)$, KPD $(\{3,4^{**}\}, 41)$ and KPD $(\{3,4^{**}\}, 35)$. Thus we obtain KPD $(\{3,4^{**}\}, 227)$ and KPD $(\{3,4^{**}\}, 233)$. A similar construction using frames of types $42^4(6x)^1$ and $36^4(6x)^1, 5 \leq x \leq 8$, solves orders $203 \leq v \leq 221$ and $179 \leq v \leq 197$. The orders $143 \leq v \leq 167 (v \neq 149)$ can be obtained from frames of types $30^4(6x)^1, 3 \leq x \leq 7 (x \neq 4)$ similarly. With the frames of type $42^4, 36^4$ and $30^4$ from Theorem 2.1 we apply Lemma 3.1 with $h = 5$ to handle the orders $173, 149$ and $125$. Also the orders $v \in \{131, 113, 95, 77\}$ come from frames of types $18^x, 7 \geq x \geq 4$. The required IKPDs and KPDs come from Lemma 2.8 and Lemma 2.10. The orders $v \in \{137, 119, 107, 101, 89, 83\}$ and $35 \leq v \leq 71$ come from Lemma 2.10.

For $v$ even, orders $32 \leq v \leq 122, v \neq 104$, and $v = 146$ come from Lemma 2.9 and Lemma 2.10. We settle the remaining orders using a similar construction as above with hole size $h = 8$ where all the needed KPDs and IKPDs from Lemma 2.8, 2.9. From Lemma 3.5 we have frames of types $24^4(6x)^1$ and $48^4(6x)^1, 4 \leq x \leq 6$, these solve orders $128 \leq v \leq 140$ and $224 \leq v \leq 236$. Frames of types $(6y)^4(6x)^1, 5 \leq y \leq 7, 4 \leq x \leq 7$, treat orders $152 \leq v \leq 218$. $v = 104$ can be obtained from a frame of type $24^4$. Now the proof is complete. □
4 Concluding remarks

We have proved Theorem 1.4. It is easy to see that $m(v) = 1$ for $v \in \{8, 11\}$ since each parallel class contains no more than three triples. An exhaustive search shows that there does not exist a KPD ($\{3, 4^{**}\}, v$) for $v = 14, 17$. A KPD ($\{3, 4^{**}\}, 23$) is given in Appendix B. Therefore, there are three values of $v \in \{20, 26, 29\}$ for which the existence of a KPD ($\{3, 4^{**}\}, v$) remains undecided.

Note added in revision (Mar. 21, 2003). The three possible exceptions $v \in \{20, 26, 29\}$ have been solved recently.

References


## Appendix A: KPD \( \{3,4^{**}\}, v \) for \( v \equiv 2 \pmod{6} \)

We construct the KPDs of orders \( 62 \leq v \leq 122 \), \( v \neq 104 \), \( v \equiv 2 \pmod{6} \) referred to in Lemma 2.10. In this case, we construct the design on the point set \( V = Z_t \times \{1, 2\} \cup H, t = (v - 6)/2, H = \{\infty_1, \ldots, \infty_6\} \). The \( (v - 6)/2 \) parallel classes will be generated mod \( t \) from an initial parallel class \( P \). For each \( v, P \) contains the following blocks.

### When \( v = 62 \):

\[
\begin{align*}
1_12_14_13_2 & \quad 1_22_24_26_1 & \quad 3_17_112_1 & \quad 5_111_118_1 & \quad 5_29_214_2 & \quad 6_212_219_2 \\
8_16_116_2 & \quad 9_119_225_2 & \quad 10_122_227_2 & \quad 13_124_20_2 & \quad 7_218_221_1 & \quad 10_220_226_1 \\
11_223_220_1 & \quad 13_221_20_1 & \quad 14_24_22 \infty_1 & \quad 15_222_2 \infty_2 & \quad 17_26_2 \infty_3 & \quad 23_115_2 \infty_4 \\
25_18_2 \infty_5 & \quad 27_117_2 \infty_6 & & & & \\
\end{align*}
\]

### When \( v = 68 \):

\[
\begin{align*}
1_12_14_11_2 & \quad 3_14_25_27_2 & \quad 28_115_219_2 & \quad 10_221_228_2 & \quad 21_125_130_1 & \quad 9_122_129_1 \\
0_{17}2_{23}2_2 & \quad 11_127_19_2 & \quad 13_18_216_2 & \quad 16_16_222_2 & \quad 5_114_214_2 & \quad 23_13_22_1 \\
7_119_10_2 & \quad 14_24_229_2 & \quad 12_118_126_1 & \quad 13_218_230_2 & \quad 6_120_2 \infty_1 & \quad 8_12_2 \infty_2 \\
10_126_2 \infty_3 & \quad 15_111_2 \infty_4 & \quad 17_125_2 \infty_5 & \quad 20_127_2 \infty_6 & & & \\
\end{align*}
\]

### When \( v = 74 \):

\[
\begin{align*}
1_12_14_11_2 & \quad 3_14_25_27_2 & \quad 31_110_218_2 & \quad 5_130_10_1 & \quad 9_121_227_2 & \quad 14_24_13_2 \\
10_121_16_2 & \quad 11_126_13_2 & \quad 13_127_18_2 & \quad 16_16_214_2 & \quad 17_120_224_2 & \quad 18_23_20_2 \\
6_{17}5_231_2 & \quad 15_122_128_2 & \quad 23_117_229_2 & \quad 8_120_228_2 & \quad 2_11_226_2 & \quad 12_25_23_2 \\
7_130_2 \infty_1 & \quad 12_122_2 \infty_2 & \quad 19_133_2 \infty_3 & \quad 25_113_2 \infty_4 & \quad 29_119_2 \infty_5 & \quad 33_116_2 \infty_6 \\
\end{align*}
\]

### When \( v = 80 \):

\[
\begin{align*}
1_12_14_11_2 & \quad 3_14_25_27_2 & \quad 18_224_20_2 & \quad 14_222_236_2 & \quad 6_130_136_1 & \quad 24_128_133_1 \\
8_120_128_2 & \quad 11_120_13_2 & \quad 13_127_18_2 & \quad 16_132_119_2 & \quad 17_134_110_2 & \quad 18_19_21_6_2 \\
22_112_27_2 & \quad 23_113_233_2 & \quad 14_123_235_2 & \quad 5_120_230_2 & \quad 25_121_232_2 & \quad 10_121_29_1 \\
9_119_125_2 & \quad 2_6_23_4_2 & \quad 7_129_2 \infty_1 & \quad 12_131_2 \infty_2 & \quad 15_126_2 \infty_3 & \quad 31_117_2 \infty_4 \\
35_115_2 \infty_5 & \quad 0_12_2 \infty_6 & & & & \\
\end{align*}
\]

### When \( v = 86 \):

\[
\begin{align*}
1_12_14_11_2 & \quad 3_14_25_27_2 & \quad 5_127_13_1 & \quad 12_120_227_2 & \quad 12_222_228_2 & \quad 32_13_21_5_2 \\
6_{17}3_23_8_2 & \quad 10_118_3_3_1 & \quad 16_16_23_5_2 & \quad 9_139_13_3_2 & \quad 8_117_26_2 & \quad 36_110_3_1_2 \\
11_124_17_2 & \quad 29_0_12_25_2 & \quad 13_120_12_3_2 & \quad 15_121_3_7_2 & \quad 28_9_2_0_2 & \quad 38_124_29_2 \\
25_13_7_2_2 & \quad 8_2_16_3_3_0_2 & \quad 14_119_13_1_5 & \quad 19_23_23_6_2 & \quad 7_134_2 \infty_1 & \quad 22_11_2 \infty_2 \\
23_121_2 \infty_3 & \quad 26_3_9_2 \infty_4 & \quad 30_118_2 \infty_5 & \quad 34_114_2 \infty_6 & & & \\
\end{align*}
\]

### When \( v = 92 \):

\[
\begin{align*}
1_12_14_11_2 & \quad 3_14_25_27_2 & \quad 36_110_230_2 & \quad 17_31_33_7_2 & \quad 15_113_220_2 & \quad 28_134_14_1_1 \\
17_225_20_2 & \quad 5_125_10_1 & \quad 28_23_24_21_4_2 & \quad 16_126_13_0_1 & \quad 20_14_21_5_2 & \quad 8_119_13_8_2 \\
14_13_8_2_1_2 & \quad 40_119_23_3_2 & \quad 32_116_4_0_2 & \quad 13_121_9_2 & \quad 31_112_22_2 & \quad 35_13_22_4_2 \\
11_114_226_2 & \quad 23_8_23_6_2 & \quad 9_118_227_2 & \quad 10_127_3_9_2 & \quad 6_122_3_7_1 & \quad 18_29_23_4_2 \\
7_142_2 \infty_1 & \quad 12_135_2 \infty_2 & \quad 24_12_2 \infty_3 & \quad 29_111_2 \infty_4 & \quad 33_123_2 \infty_5 & \quad 39_16_2 \infty_6 \\
\end{align*}
\]
$v = 98: \quad 1_{12}4_{12}1_2 \quad 3_{12}4_{22}5_{22}7_2 \quad 13_{12}4_{41}30_2 \quad 36_{12}9_{22}20_2 \quad 0_{12}3_{22}34_2 \quad 15_{12}4_{51}24_2$
$2_{27}3_{12}4_{38_1} \quad 3_{27}4_{42}2_{27_2} \quad 16_{27}1_{032}3_6_2 \quad 25_{27}3_{52}4_{02_2} \quad 18_{27}1_{32}3_{12_2} \quad 11_{27}1_{92}2_{81_2}$
$2_{22}3_{41}1_{12} \quad 6_{22}1_{22}3_{01_2} \quad 39_{22}1_{42}4_{24_2} \quad 7_{22}1_{92}2_{32_2} \quad 10_{22}3_{92}3_{72_2} \quad 8_{22}1_{02}4_{01_2}$
$1_{14}1_{24}4_{22_2} \quad 2_{14}3_{41}4_{52_2} \quad 9_{14}5_{23}2_{32_2} \quad 12_{14}2_{12}2_{02_2} \quad 2_{31}6_{23}1_{6_2} \quad 17_{22}5_{23}9_{22}2$
$3_{35}1_{26}3_{32_2} \quad 2_{22}2_{82}2_{41_2} \quad 5_{132}3_{82}1 \quad 17_{143}2_{32_2} \quad 2_{61}1_{82}2_{32_2} \quad 3_{21}8_{22}2_{44_2}$
$3_{33}2_{92}2_{65_2} \quad 4_{132}1_{26_2}$

$v = 110: \quad 1_{12}4_{12}1_2 \quad 3_{12}4_{22}5_{22}7_2 \quad 2_{81}2_{02}4_{92}2_2 \quad 8_{14}8_{22}2_{82_2} \quad 3_{01}4_{12}1_{49_2} \quad 14_{22}1_{22}4_{22_2}$
$2_{31}4_{71}2_{12} \quad 1_{61}4_{41}2_{25_2} \quad 4_{61}3_{22}2_{36_2} \quad 6_{12}2_{92}3_{31_2} \quad 12_{10}1_{52}3_{02_2} \quad 2_{22}1_{52}2_{48_2}$
$1_{18}3_{72}4_{72_2} \quad 5_{21}6_{21}1_{32_2} \quad 2_{13}2_{82}3_{42_2} \quad 13_{23}1_{32}4_{02_2} \quad 9_{41}3_{22}7_{27_2} \quad 10_{15}1_{52}0_{22_2}$
$2_{27}3_{17}4_{32_2} \quad 7_{22}2_{22}2_{31_2} \quad 2_{31}3_{82}3_{32_2} \quad 2_{24}1_{42}4_{52_2} \quad 2_{51}2_{23}3_{39_2} \quad 3_{41}1_{02}5_{21_2}$
$1_{19}3_{21}3_{91_2} \quad 3_{61}4_{51}1_{51_2} \quad 3_{51}1_{72}3_{02_2} \quad 2_{62}1_{24}2_{46_2} \quad 9_{22}2_{62}4_{41_2} \quad 16_{24}2_{42}3_{42_2}$
$1_{11}1_{02}2_{11_2} \quad 1_{41}4_{42}2_{24_2} \quad 2_{17}2_{92}2_{21_2} \quad 2_{13}1_{82}2_{32_2} \quad 4_{21}8_{12}2_{21_2}$
$3_{40}1_{12}2_{15_2} \quad 3_{51}1_{22}2_{66_2}$

$v = 116: \quad 1_{12}4_{12}1_2 \quad 3_{12}4_{22}5_{22}7_2 \quad 2_{51}3_{22}2_{11_2} \quad 2_{31}9_{22}2_{42_2} \quad 4_{92}2_{82}4_{20_2} \quad 3_{10}5_{31}3_{2
$v = 122: \quad 1_{12}4_{12}1_2 \quad 3_{12}4_{22}5_{22}7_2 \quad 5_{51}7_{11}2_{43_2} \quad 9_{12}7_{72}3_{62_2} \quad 2_{51}3_{22}4_{24_2} \quad 4_{92}2_{42}5_{52_2}$
$2_{20}4_{51}5_{51_2} \quad 3_{20}1_{92}5_{02_2} \quad 4_{21}6_{21}2_{12_2} \quad 1_{41}3_{31}4_{72_2} \quad 1_{91}7_{22}0_{22_2} \quad 3_{22}5_{31}2_{32_2}$
$6_{13}7_{51}4_{11_2} \quad 2_{13}1_{52}2_{92_2} \quad 3_{22}3_{51}4_{92_2} \quad 3_{61}4_{81}2_{26_2} \quad 2_{21}4_{01}4_{41_2} \quad 3_{11}4_{12}2_{56_2}$
$2_{29}3_{51}9_{2_2} \quad 2_{26}1_{02}4_{52_2} \quad 1_{01}8_{31}3_{32_2} \quad 7_{12}2_{57_2} \quad 1_{31}4_{11}2_{42_2} \quad 2_{30}3_{24}3_{42_2}$
$2_{22}1_{46}2_{82_2} \quad 5_{12}5_{22}4_{02_2} \quad 1_{61}2_{32}3_{81_2} \quad 8_{21}2_{51_2} \quad 1_{51}2_{47}5_{56_2} \quad 5_{01}8_{23}1_{62_2}$
$6_{54}2_{22}2_{72_2} \quad 1_{32}3_{22}5_{32_2} \quad 1_{82}3_{12}3_{28_2} \quad 2_{32}3_{22}4_{82_2} \quad 2_{11}2_{22}2_{24_2} \quad 1_{71}5_{42}2_{62_2}$
$2_{25}1_{20}2_{33_2} \quad 3_{41}5_{52}2_{44_2} \quad 3_{91}4_{42}2_{52_2} \quad 4_{31}1_{42}2_{66_2}$

Appendix B: KPD ($\{3,4^{**}\}, v$) for $v \equiv 5 \pmod{6}$

We construct a KPD ($\{3,4^{**}\},23$) referred to in Lemma 2.10 on the point set $V = Z_9 \times \{1,2\} \cup H$, $H = \{a\} \times Z_3 \cup \{\infty_1,\infty_2\}$. The required nine parallel classes will be generated mod 9 from the initial parallel classes $P$. $P$ contains the following blocks where the subscripts on $a$ are evaluated mod 3.

$1_{12}4_{12}1_2 \quad 2_{23}2_{36}2_{61} \quad 3_{18}1_{a0} \quad 1_{28}2_{a1} \quad 0_{17}2_{a2} \quad 5_{14}2_{32_2} \quad 7_{102_2}$

The KPD ($\{3,4^{**}\},41$) referred to in Lemma 2.10 is constructed on the point set $V = Z_9 \times \{1,2,3,4\} \cup H$, $H = \{\infty_1,\ldots,\infty_5\}$. The required 18 parallel classes will be generated mod 9 from two initial parallel classes $P_1$ and $P_2$. The blocks in $P_1$ and $P_2$ are as follow.
The KPD \( \{3, 4^*, 5\} \) referred to in Lemma 2.10 is constructed on the point set \( V = Z_{15} \times \{1, 2, 3, 4\} \cup H \), where \( H = \{\infty_1, \ldots, \infty_5\} \). The required 30 parallel classes will be generated mod 15 from two initial parallel classes \( P_1 \) and \( P_2 \). The blocks in \( P_1 \) and \( P_2 \) are as follow.

\[
\begin{align*}
P_1 & : \quad 1_{2}1_{4}1_{2} \quad 3_{1}4_{2}5_{2}1_{3} \quad 5_{1}0_{2}2_{4} \quad 6_{1}5_{3}6_{3} \quad 7_{1}1_{4}2_{4} \quad 8_{1}4_{6}4_{4} \\
& \quad 2_{3}3_{9}0_{3} \quad 3_{7}3_{3}4 \quad 6_{2}8_{3}\infty \_1 \quad 7_{2}0_{4}\infty \_2 \quad 8_{2}5_{4}\infty \_3 \quad 0_{9}8_{4}\infty \_4 \\
& \quad 4_{3}7_{4}\infty \_5 \\
\end{align*}
\[
\begin{align*}
P_2 & : \quad 1_{2}3_{2}4_{8} \quad 2_{3}4_{3}1_{4}6_{4} \quad 1_{8}2_{3}4_{1} \quad 8_{1}2_{2}5_{4} \quad 0_{1}4_{2}0_{4} \quad 6_{1}7_{3}7_{4} \\
& \quad 3_{5}2_{5}0_{2} \quad 6_{2}5_{3}0_{3} \quad 5_{1}4_{4}\infty \_1 \quad 3_{1}6_{3}\infty \_2 \quad 4_{1}8_{3}\infty \_3 \quad 7_{1}3_{3}\infty \_4 \\
& \quad 2_{7}\infty \_5 \\
\end{align*}
\]

We construct the KPDs of orders \( v \in \{35, 47, 59, 71, 83, 107, 119\} \), \( v \equiv 5 \pmod{6} \) referred to in Lemma 2.10. In this case, we construct the design on the point set \( V = Z_{t} \times \{1, 2\} \cup H, t = (v - 5)/2 \), \( H = \{\infty_1, \ldots, \infty_5\} \). The \((v - 5)/2\) parallel classes will be generated mod \( t \) from an initial parallel class \( P \). For each \( v, P \) contains the following blocks.

\[
\begin{align*}
v = 35 & : \quad 1_{2}1_{4}3_{2} \quad 1_{2}2_{4}2_{7}3 \quad 3_{1}8_{1}2_{12} \quad 5_{2}10_{2}14_{2} \quad 5_{1}13_{1}9_{2} \quad 8_{2}0_{2}10_{1} \\
& \quad 6_{1}3_{2}\infty \_1 \quad 9_{1}2_{2}\infty \_2 \quad 1_{1}1_{11}\infty \_3 \quad 1_{4}1_{2}\infty \_4 \quad 0_{1}6_{2}\infty \_5 \\
\end{align*}
\[
\begin{align*}
v = 47 & : \quad 1_{2}1_{2}3_{2} \quad 2_{5}2_{7}1_{9} \quad 3_{1}6_{1}0_{1} \quad 4_{1}1_{4}1_{19} \quad 4_{2}8_{2}16_{2} \quad 1_{2}2_{13}2_{19}2 \\
& \quad 5_{1}3_{1}1_{17}2 \quad 8_{1}1_{7}1_{14}2 \quad 1_{0}2_{0}1_{61}2 \quad 1_{5}2_{0}2_{12}1 \quad 1_{1}1_{18}\infty \_1 \quad 1_{5}1_{7}\infty \_2 \\
& \quad 1_{8}1_{6}\infty \_3 \quad 2_{0}1_{9}\infty \_4 \quad 0_{1}1_{12}\infty \_5 \\
\end{align*}
\[
\begin{align*}
v = 59 & : \quad 1_{2}4_{4}3_{2} \quad 1_{2}2_{4}2_{6}1 \quad 3_{1}7_{1}2_{12} \quad 5_{1}1_{1}1_{18}1 \quad 5_{2}9_{2}14_{2} \quad 6_{2}1_{2}1_{29}2 \\
& \quad 8_{1}6_{1}2_{0}2 \quad 9_{1}1_{9}2_{22}2 \quad 1_{0}2_{2}1_{15}2 \quad 1_{3}2_{4}2_{12}1 \quad 1_{3}2_{4}2_{17}2 \quad 1_{0}2_{18}2_{26}1 \\
& \quad 1_{1}2_{2}2_{0}1 \quad 1_{6}2_{0}2_{0}1 \quad 1_{4}1_{2}3_{2}\infty \_1 \quad 1_{5}1_{2}5\infty \_2 \quad 2_{1}8_{2}\infty \_3 \quad 2_{3}1_{7}\infty \_4 \\
& \quad 2_{5}1_{3}\infty \_5 \\
\end{align*}
\[
\begin{align*}
v = 71 & : \quad 1_{2}1_{1}3_{2} \quad 2_{8}2_{9}2_{3}1_{51} \quad 7_{1}0_{1}1_{14}1 \quad 1_{7}2_{2}1_{28}1 \quad 1_{1}1_{19}2_{129}1 \quad 2_{1}2_{4}2_{30}2 \\
& \quad 1_{6}2_{1}2_{3}2 \quad 5_{2}1_{0}2_{18}2 \quad 1_{2}1_{2}1_{62}2 \quad 4_{1}6_{1}2_{2} \quad 6_{1}1_{0}1_{2}9 \quad 1_{3}3_{0}1_{17}2 \\
& \quad 1_{8}1_{3}1_{27}2 \quad 4_{2}2_{0}2_{23}1 \quad 7_{2}2_{2}2_{27}1 \quad 8_{2}3_{1}2_{25}1 \quad 1_{3}2_{5}2_{8}1 \quad 1_{4}2_{0}2_{61} \\
& \quad 9_{1}1_{9}\infty \_1 \quad 1_{5}2_{6}\infty \_2 \quad 2_{4}1_{3}2\infty \_3 \quad 3_{2}1_{11}\infty \_4 \quad 0_{1}1_{5}\infty \_5 \\
\end{align*}
\]
\[ v = 83 : \begin{array}{cccccccccc}
1 & 2 & 4 & 1 & 12 & 3 & 2 & 4 & 15 & 71
\end{array} \]

\[ 7 & 1 & 22 & 29 & 1 & 25 & 1 & 14 & 2 & 32
\]

\[ 20 & 26 & 2 & 35 & 1 & 11 & 2 & 92
\]

\[ 24 & 1 & 2 & 81 & 34 & 1 & 14 & 1 & 7 & 02
\]

\[ 8 & 1 & 27 & 1 & 38 & 1 & 61 & 18 & 31 & 1
\]

\[ 32 & 8 & 1 & 82 & 1 & 82
\]

\[ 32 & 1 & 62 & 10 & 2
\]

\[ 5 & 1 & 23 & 2 & 37 & 2
\]

\[ 19 & 1 & 27 & 2 & 33 & 2 & 26 & 1 & 20 & 31 & 2
\]

\[ 12 & 1 & 22 & 3 & 32 & 82 & 15 & 1 & 11 & 2 & 42
\]

\[ 16 & 1 & 21 & 37 & 1
\]

\[ 9 & 1 & 17 & 1 & 36 & 2
\]

\[ 23 & 1 & 15 & 2 & 34 & 2 & 33 & 1 & 16 & 2 & 82
\]

\[ 36 & 1 & 13 & 2 & 21 & 2 & 10 & 1 & 30 & 2 & \infty 1
\]

\[ 13 & 1 & 25 & 1 & 2 & \infty 2
\]

\[ 30 & 1 & 12 & 2 & \infty 3 & 35 & 1 & 19 & 2 & \infty 4
\]

\[ 0 & 1 & 29 & 2 & \infty 5
\]

\[ v = 107 : \begin{array}{cccccccccc}
1 & 2 & 4 & 1 & 12 & 45 & 1 & 82 & 24 & 2 & 02
\end{array} \]

\[ 35 & 1 & 92 & 40 & 2 & 25 & 1 & 37 & 1 & 01 & 47 & 1 & 42 & 1 & 82
\]

\[ 15 & 1 & 39 & 1 & 46 & 2
\]

\[ 23 & 1 & 35 & 2 & 50 & 2 & 16 & 2 & 27 & 2 & 30 & 2
\]

\[ 9 & 1 & 43 & 1 & 72 & 8 & 1 & 31 & 41 & 2 & 11 & 1 & 27 & 1 & 49 & 1
\]

\[ 16 & 1 & 2 & 25 & 2
\]

\[ 14 & 1 & 32 & 1 & 10 & 2 & 36 & 2 & 38 & 2 & 43 & 2
\]

\[ 19 & 1 & 28 & 1 & 45 & 2 & 21 & 4 & 41 & 3 & 41 & 2 & 61 & 4 & 61 & 47 & 2
\]

\[ 10 & 1 & 41 & 1 & 29 & 2
\]

\[ 29 & 1 & 20 & 2 & 32 & 2 & 34 & 1 & 42 & 4 & 12 & 12 & 1 & 6 & 12 & 12 & 2
\]

\[ 12 & 1 & 5 & 2 & 3 & 3 & 0 & 3 & 2 & 2 & 2 & 20 & 3 & 4 & 2 & 4 & 4 & 4 & 2
\]

\[ 40 & 1 & 34 & 2 & 4 & 4 & 2
\]

\[ 18 & 1 & 26 & 1 & 33 & 1 & 31 & 1 & 24 & 2 & 12 & 1 & 5 & 15 & 2 & 38 & 1 & 42 & 1 & 48 & 1
\]

\[ 11 & 2 & 2 & 82 & 3 & 72 & 19 & 2 & 4 & 82 & 4 & 92
\]

\[ 5 & 1 & 13 & 2 & \infty 1 & 7 & 1 & 39 & 2 & \infty 2 & 20 & 1 & 33 & 2 & \infty 3 & 22 & 1 & 42 & 2 & \infty 4 & 36 & 1 & 21 & 2 & \infty 5
\]

\[ v = 119 : \begin{array}{cccccccccc}
1 & 2 & 4 & 1 & 12 & 3 & 2 & 4 & 15 & 71
\end{array} \]

\[ 14 & 1 & 37 & 1 & 10 & 2 & 11 & 2 & 38 & 2 & 44 & 2 & 29 & 1 & 42 & 4 & 72 & 2 & 9 & 1 & 37 & 2 & 53
\]

\[ 30 & 1 & 37 & 2 & 53 & 2
\]

\[ 10 & 1 & 0 & 35 & 2 & 18 & 1 & 26 & 1 & 54 & 1 & 16 & 1 & 49 & 1 & 31 & 2 & 28 & 1 & 40 & 1 & 23 & 2 & 11 & 1 & 38 & 1 & 25 & 2 & 9 & 1 & 44 & 1 & 50 & 1
\]

\]

\[ 20 & 1 & 31 & 4 & 12 & 27 & 3 & 39 & 5 & 02 & 32 & 2 & 4 & 2 & 43 & 2 & 7 & 1 & 36 & 5 & 34 & 2 & 23 & 1 & 8 & 2 & 40 & 2 & 8 & 1 & 48 & 1 & 34 & 2
\]

\[ 34 & 1 & 2 & 42 & 56 & 2 & 13 & 1 & 27 & 1 & 46 & 2 & 46 & 1 & 20 & 3 & 02 & 3 & 2 & 16 & 2 & 52 & 2 & 5 & 1 & 29 & 2 & \infty 1 & 6 & 1 & 15 & 2 & \infty 2
\]

\[ 22 & 1 & 49 & 2 & \infty 3 & 35 & 1 & 26 & 2 & \infty 4 & 53 & 1 & 92 & \infty 5
\]