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1 Introduction

Most graph labeling methods trace their origin to one introduced by Rosa [452] in 1967, or one given by Graham and Sloane [249] in 1980. Rosa [452] called a function f a *β-valuation* of a graph G with q edges if f is an injection from the vertices of G to the set $\{0, 1, \ldots, q\}$ such that, when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct. Golomb [243] subsequently called such labelings *graceful* and this is now the popular term. Rosa introduced β-valuations as well as a number of other labelings as tools for decomposing the complete graph into isomorphic subgraphs. In particular, β-valuations originated as a means of attacking the conjecture of Ringel [447] that K_{2n+1} can be decomposed into $2n + 1$ subgraphs that are all isomorphic to a given tree with n edges. Although an unpublished result of Erdős says that most graphs are not graceful (cf. [249]), most graphs that have some sort of regularity of structure are graceful. Sheppard [502] has shown that there are exactly $q!$ gracefully labeled graphs with q edges. Balakrishnan and Sampathkumar [68] have shown that every graph is a subgraph of a graceful graph. Rosa [452] has identified essentially three reasons why a graph fails to be graceful: (1) G has "too many vertices" and "not enough edges", (2) G "has too many edges", and (3) G "has the wrong parity". An infinite class of graphs that are not graceful for the second reason is given in [103]. As an example of the third condition Rosa [452] has shown that if every vertex has even degree and the number of edges is congruent to 1 or 2 (mod 4) then the graph is not graceful. In particular, the cycles C_{4n+1} and C_{4n+2} are not graceful.

Harmonious graphs naturally arose in the study by Graham and Sloane [249] of modular versions of additive bases problems stemming from error-correcting codes. They defined a graph G with q edges to be *harmonious* if there is an injection f from the vertices of G to the group of integers modulo q such that when each edge xy is assigned the label $f(x) + f(y) \pmod{q}$, the resulting edge labels are distinct. When G is a tree, exactly one label may be used on two vertices. Analogous to the "parity" necessity condition for graceful graphs, Graham and Sloane proved that if a harmonious graph has an even number q of edges and the degree of every vertex is divisible by 2^k then q is divisible by 2^{k+1}. Thus, for example, a book with seven pages (i.e., the cartesian product of the complete bipartite graph $K_{1,7}$ and a path of length 1) is not harmonious. Liu and Zhang [388] have generalized this condition as follows: if a harmonious graph with q edges has degree sequence $d_1, d_2, \ldots, d_p$ then $\gcd(d_1, d_2, \ldots d_p, q)$ divides $q(q-1)/2$. They have also proved that every graph is a subgraph of a harmonious graph.

Over the past three decades in excess of 600 papers have spawned a bewildering array of graph labeling methods. Despite the unabated procession of papers, there are few general results on graph labelings. Indeed, the papers focus on particular classes of graphs and methods, and feature ad hoc arguments. In part because many of the papers have appeared in journals not widely available, frequently the same classes of graphs have been done by several authors. In this article, we survey what is known about numerous graph labeling methods. The author requests that he be sent preprints and reprints as well as corrections for inclusion in the updated versions of the survey.

Earlier surveys, restricted to one or two labeling methods, include [91], [106], [318], [229], and [231]. The extension of graceful labelings to directed graphs arose in the characterization of finite neofields by Hsu and Keedwell [288], [289]. The relationship between graceful digraphs and a variety of algebraic structures including cyclic difference sets, sequenceable groups, generalized complete mappings, near-complete mappings and neofields is discussed in [109], [110]. The connection between graceful labelings and perfect systems of difference sets is given in [94]. Labeled graphs serve as useful models for a broad range of applications such as: coding theory, x-ray crystallography, radar, astronomy, circuit design, communication network addressing and data base management–see [107], [108] and [549] for details. Terms and notation not defined below follow that used in [155] and [229].

2 Graceful and Harmonious Labelings

2.1 Trees

The Ringel-Kotzig conjecture that all trees are graceful has been the focus of many papers. Kotzig [291] has called the effort to prove it a "disease." Among the trees known to be graceful are: caterpillars [452] (a *caterpillar* is a tree with the property that the removal of its endpoints leaves a path); trees with at most 4 end-vertices [291], [616] and [298]; trees with diameter at most 5 [616] and [286]; trees with at most 27 vertices [21]; symmetrical trees (i.e., a rooted tree in which every level contains vertices of the same degree) [95], [439]; rooted trees where the roots have odd degree and the lengths of the paths from the root to the leaves differ by at most one and all the internal vertices have the same parity [145]; regular bamboo trees [462] (a rooted tree consisting of branches of equal length the endpoints of which are identified with end points of stars of equal size; and olive trees [433] and [2] (a rooted tree consisting of k branches, where the ith branch is a path of length i). Aldred, Širáň and Širáň [22] have proved that the number of graceful labelings of P_n grows at least as fast as $(5/3)^n$. They mention that this fact has an application to topological graph theory. Stanton and Zarnke [542] and Koh, Rogers and Tan [319] gave methods for combining graceful trees to yield larger graceful trees. Burzio and Ferrarese [133] have shown that the graph obtained from any graceful tree by subdividing every edge is also graceful. Morgan [415] has used Skolem sequences to construct classes of graceful trees. In 1979 Bermond [91] conjectured that lobsters are graceful (a *lobster* is a tree with the property that the removal of the endpoints leaves a caterpillar). Special cases of this conjecture have been done by Ng [426], by Wang, Jin, Lu and Zhang [579] and by Abhyanker [1]. Morgan [414] has shown that all lobsters with perfect matchings are graceful. Morgan and Rees [416] have used Skolem and Hooked-Skolem sequences to generate classes of graceful lobsters. Whether or not lobsters are harmonious seems to have attracted no attention thus far. Chen, Lü and Yeh [157] define a *firecracker* as a graph obtained from the concatenation of stars by linking one leaf from each. They also define a *banana tree* as a graph obtained by connecting a vertex v to one leaf of each of any number of stars (v is not in any of the stars). They proved that firecrackers are graceful and conjecture that banana trees are graceful. Various kinds of bananas trees have been shown to be graceful by Bhat-Nayak and Deshmukh [99], by Murugan and Arumugam [420], [422] and by Vilfred [570]. Despite the efforts of many, the graceful tree conjecture remains open even for trees with maximum degree 3. Aldred and McKay [21] used a computer to show that all trees with at most 26 vertices are harmonious. That caterpillars are harmonious has been shown by Graham and Sloane [249]. Cahit extended the notion of gracefulness to directed graphs in [146]. More specialized results about trees are contained in [91], [106], [318], [390], [140] and [297].